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Large-N expansion of the 2-matrix model

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Abstract: We present a method, based on loop equations, to compute recursively, all the terms in the large N topological expansion of the free energy for the 2-hermitean matrix model. We illustrate the method by computing the first subleading term, i.e. the free energy of a statistical physics model on a discretized torus.

Keywords: 1/N Expansion, 2D Gravity, Matrix Models.

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Contents

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1. Introduction

Random matrix models [1]–[6] have a wide range of applications in mathematics and physics where they constitute a major field of activity. They are involved in condensed matter physics (quantum chaos [4, 7], localization, crystal growths [8],... etc), statistical physics [3, 9, 10, 11] (on a 2d fluctuating surface, also called 2d euclidean quantum gravity, linked to conformal field theory), high energy physics (string theory [12], quantum gravity [3, 11, 13], QCD [14],...), and they are very important in mathematics too: (they seem to be linked to the Riemann conjecture [1, 15]), they are important in combinatorics, and provide a wide class of integrable systems [5, 16, 17].

In the 80's, random matrix models were introduced as a toy model for zero–dimensional string theory and quantum gravity [9, 3, 10]. By fine-tuning the parameters of the model, one can reach many multi-critical points, which are in relationship with the (p, q) minimal conformal models [18, 19]. The 1-hermitean matrix models is limited to $(p, 2)$ conformal minimal models, whereas the 2-matrix model can represent any (p, q) conformal minimal model with $q \geq 2$ [19]. The 2-matrix model is thus more general than the 1-matrix model, and it has been the source of considerable interest in the past few months [20, 17, 21]–[27].

The free energy of the 1-matrix model was conjectured [28, 3, 29, 30] and also rigorously proven [31] to have a $1/N^2$ expansion called topological expansion (N is the size of the matrix):

$$F = \sum_{h=0}^{\infty} N^{-2h} F^{(h)} \,. \tag{1.1}$$

The authors of [29] invented an efficient method to compute recursively all the $F^{(h)}$'s, and they improved it in [30].

The 2-matrix model is conjectured to have a similar $1/N^2$ topological expansion. Indeed, this expansion is the main motivation for applications to 2-dimensional quantum gravity [3, 2], because each $F^{(h)}$ is the partition function of a statistical physics model on a genus h surface. There is at the present time no rigorous proof[1] of the existence of such an expansion, but, assuming that it exists, the aim of the present work is to give a method to compute recursively the terms of the expansion, similar to that of [29].

The 2-matrix model was first introduced as a model for two-dimensional gravity, with matter, and in particular with an Ising field [33, 34]. The diagrammatic expansion of the 2-matrix model's partition function is known to generate 2-dimensional statistical physics models on a random discrete surface [3, 9, 33]:

$$N^2 F = -\ln Z = \sum_{\text{surfaces}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} \tag{1.2}$$

where the Action is the matter action (like Ising's nearest neighboor spin coupling) plus the gravity action (total curvature and cosmological constant) [3]. The cosmological constant couples to the area of the surface, and N (the size of the matrix) couples to the total curvature, i.e. the genus of the surface. The large-N expansion thus generates a genus expansion:

$$F = \sum_{h=0}^{\infty} N^{-2h} F^{(h)} \tag{1.3}$$

where $F^{(h)}$ is the partitrion function of the statistical physics model on a random surface of fixed genus h.

$$F^{(h)} = \sum_{\text{genus } h \text{ surfaces}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} . \tag{1.4}$$

The leading term $F^{(0)}$ is the planar contribution. It can be computed by many different methods, for instance the saddle point method (along the method invented by [40] and rigorously established by [26]) or the loop equation method which we will explain below. Our goal in this article is to compute $F^{(1)}$ and present an algorithmic method for computing $F^{(h)}$ for $h \geq 1$. We generalize the method of [29].

1.1 Reminder: the one-matrix model

For a given polynomial $V(x)$ of degree $d+1$, called the potential, and a given integer N, we define the partition function Z and the free energy F as:

$$Z = \mathrm{e}^{-N^2 F} = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)} \tag{1.5}$$

where the integral is over the set of hermitean matrices of size N, with the measure $\mathrm{d}M$ equal to the product of Lebesgue measures of all real components of M.

The free energy has a $1/N^2$ power series expansion (under the 1-cut asumption, this will be discussed in more details in section 4.1):

$$F = \sum_{h=0}^{\infty} N^{-2h} F^{(h)} . \tag{1.6}$$

[1] The Riemann-Hilbert approach seems to be the best way to prove the existence of the $1/N^2$ topological extension as in [31]. The Riemann-Hilbert problem for the 2-matrix model has been formulated [22, 25, 24, 32], and seems to be on the verge of being solved [24].

In their pioneering work, the authors of [29] invented a method to compute recursivelly all the $F^{(h)}$'s. In particular they found that the genus one free energy for the 1-matrix model is:

$$F^{(1)} = -\frac{1}{24} \ln \left((b-a)^4 M(a) M(b) \right) \tag{1.7}$$

where b, a and the polynomial $M(x)$ are related to the large-N limit of the average density of eigenvalues of the matrix:

$$\rho(x) = \frac{1}{2\pi} M(x) \sqrt{(x-a)(b-x)}\,, \qquad M(x) = \mathop{\mathrm{Pol}}_{x\to\infty} \frac{V'(x)}{\sqrt{(x-a)(x-b)}} \tag{1.8}$$

This density has a compact support $[a, b]$, and $M(x)$ is a polynomial of degree $d-1$ (the 1-matrix case is re-explained in section 7).

We are going to extend this kind of expression (eq. (1.7)) for the 2-matrix model.

1.2 Outline of the article

- In section 2 we introduce the definitions and notations, in particular we define the 1-loop functions and 2-loop functions, and the loop-insertion operators.
- In section 3 we explain the loop equation method, and derive the "Master loop equation".
- In section 4, we observe that, to leading order, the master loop equation is an algebraic equation of genus zero, and we study the geometry and the sheet-structure of the underlying algebraic curve.
- In section 5 we use the algebraic master loop equation to compute all the previously defined loop functions to leading order.
- In section 6 we include the previously neglected $1/N^2$ term in the loop equation, and we compute the 1-loop function $Y(x)$ to next to leading order. Then we derive the next to leading order free energy $F^{(1)}$ by integrating $Y^{(1)}(x)$. We also discuss how to compute higher order terms.
- In section 7 we compare with the results previously known for the one matrix case.
- section 8 is the conclusion.

2. The 2-matrix model

2.1 The partition function

For two given polynomials $V_1(x)$ of degree d_1+1 and $V_2(y)$ of degree d_2+1, called the potentials, and a given integer N, we define the partition function Z and the free energy F as:

$$Z = \mathrm{e}^{-N^2 F} = \int \mathrm{d}M_1 \mathrm{d}M_2\, \mathrm{e}^{-N\,\mathrm{tr}[V_1(M_1)+V_2(M_2)-M_1M_2]} \tag{2.1}$$

where the integral is over the set of pairs of hermitean matrices M_1 and M_2 of size N, with the measure $\mathrm{d}M_1\mathrm{d}M_2$ equal to the product of Lebesgue measures of all real components of M_1 and M_2 (see [3] for definitions).

The potentials V_1 and V_2 are written:

$$V_1(x) = \sum_{k=0}^{d_1} \frac{g_{k+1}}{k+1} x^{k+1}\,, \qquad V_2(y) = \sum_{k=0}^{d_2} \frac{g^*_{k+1}}{k+1} y^{k+1} \tag{2.2}$$

Remark: eq. (2.1) is well defined only when the potentials are bounded from below, i.e. when:

$$\left\{ d_1 \text{ and } d_2 \text{ are odd} \quad \operatorname{Re} g_{d_1+1} > 0\,,\ \operatorname{Re} g^*_{d_2+1} > 0 \right\}. \tag{2.3}$$

However, it is known [3] how to give a meaning to (2.1) for potentials not bounded from below, in the large-N limit (somehow the tunnel effect allowing the matrices to "escape from the potential wells" is suppressed like $O(\mathrm{e}^{-N})$ in the large-N limit, it plays no role in the $1/N^2$ expansion). Therefore, we will consider arbitrary polynomial potentials which do not necessarily satisfy condition (2.3). For simplicity, we will assume that the potentials V_1 and V_2 are real.[2]

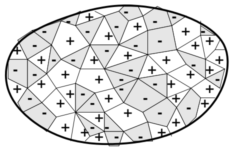
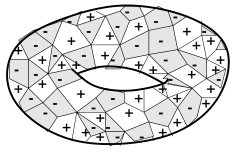
In addition, we will assume that the potentials V_1 and V_2 are generic, i.e. they are not critical (we will explain what is a critical potential in section 4.4.1, see [19]).

The partition function F can be interpreted (through the Feynman graphs expansion, see [3, 2]) as the generating function of surfaces made of polygons carrying spins: spin "up" (+) for the polygons generated by the vertices of M_1, "down" (-) for the polygons generated by the vertices of M_2.

$$F = \sum_{S\in\mathcal{E}} \frac{N^{-2h(S)}}{\#\operatorname{Aut}(S)} \left(\frac{g_2}{g_2 g_2^* - 1}\right)^{n_{++}} \left(\frac{g_2^*}{g_2 g_2^* - 1}\right)^{n_{--}} \prod_{k=1,k\neq 2}^{d_1+1} (-g_k)^{n_k(S)} \prod_{j=1,j\neq 2}^{d_2+1} (-{g^*}_j)^{\tilde{n}_j(S)} \tag{2.4}$$

where $\mathcal{E}$ is the ensemble of surfaces made of up-polygons with at most d_1+1 sides and down-polygons with at most d_2+1 sides. $\operatorname{Aut}(S)$ is the group of automorphisms of the graph of edges of S, $h(S)$ is the genus of S, $n_k(S)$ $(1 \leq k \leq d_1+1)$ is the number of up k-gons of S, $\tilde{n}_j(S)$ $(1 \leq j \leq d_2+1)$ is the number of down j-gons of S, and n_{++} (resp. n_{--}) is the number of edges separating two up polygons (resp. two down polygons).

We write digrammatically:

$F =$  $+ \dfrac{1}{N^2}$  $+ \cdots .$

This is why the two-matrix model can represent the Ising model on a random discretized surface [33, 34], and other 2d statistical physics models on a random surface

[2]The derivation of the loop equations in section 3 should be slightly modified in the case of non-real potentials, because some change of variables we consider would be no longer hermitean. But most of the results derived in this article are still valid for complex potentials.

carrying some type of matter [3]. It is clear from eq. (2.4) that relevant applications to 2d statistical physics on a random surface involve potentials which violate condition 2.3 (all the g_k's and $\tilde{g}_j$'s must be negative in order to have positive weights).

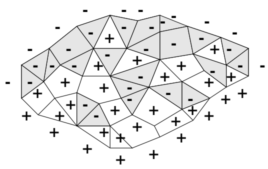
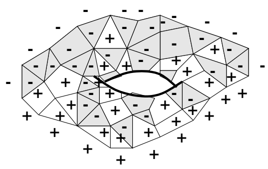
2.2 Definition of the loop functions

We introduce the following expectation values computed with the probability weight $\frac{1}{Z}\mathrm{e}^{-N\,\mathrm{tr}[V_1(M_1)+V_2(M_2)-M_1 M_2]}$:

$$T_{k,l} = \frac{1}{N}\left\langle \mathrm{tr}\, M_1^k M_2^l \right\rangle \tag{2.5}$$

We will be particularly interested in their large-N limits and large-N expansion.

$T_{k,l}$ can be interpreted as the partition function of a statistical physics model on surfaces with one boundary of lenght $k+l$ (k contiguous $+$ and j contiguous $-$):

$T_{k,l} =$  $+ \frac{1}{N^2}$  $+ \cdots .$

It is more convenient to introduce generating functions which contain all the T_{kl}'s at once:

2.2.1 One-loop functions

We define the following formal functions of the complex variables x and y:

$$W(x) = \frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{x-M_1}\right\rangle = \sum_{k=0}^{\infty}\frac{T_{k,0}}{x^{k+1}}, \qquad Y(x) = V_1'(x) - W(x) \tag{2.6}$$

$$\tilde{W}(y) = \frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{y-M_2}\right\rangle = \sum_{k=0}^{\infty}\frac{T_{0,k}}{y^{k+1}}, \qquad X(y) = V_2'(y) - \tilde{W}(y)\,. \tag{2.7}$$

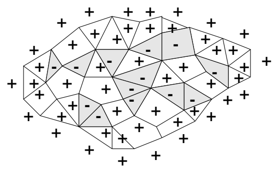
These functions are formally defined only through their large x and large y expansions, they are only a convenient rewritting of the collection of $T_{k,0}$ and $T_{0,k}$. The sum is not necessarily convergent.

We will find below, that $W^{(0)}(x) = \lim_{N\to\infty} W(x)$ (resp. $\tilde{W}(y)$) satisfies an algebraic equation, and thus $W^{(0)}(x)$ (resp. $\tilde{W}^{(0)}(y)$) is an analytical function of x (resp. y), in the complex plane, with a cut. The radius of convergence of eq. (2.6) is finite in the large-N limit. Their analytical structure will be dicussed in details in part 4.4.

The functions $W(x)$ and $\tilde{W}(y)$ are called the resolvents, they provide most of the information about the statistical properties of the spectra of M_1 and M_2. For instance the location of the cut of $W^{(0)}(x)$, is the support of the large-N average density of eigenvalues of M_1, and in the large-N limit, the average density of eigenvalues is proportional to the discontinuity of $W^{(0)}(x)$ along its cuts.

Their diagrammatic expansion is the generating function for statistical physics models coupled to gravity, on a surface with one boundary (a disc if genus zero), this is why they are called one-loop functions.

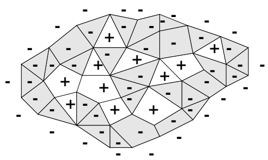
$$W(x) = \sum_{l_1=0}^{\infty} x^{-l_1-1} \sum_{\substack{\text{surfaces,} \\ \text{boundary} = l_1 \text{ up}}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} = $$



$$+ \cdots \tag{2.8}$$

the sum carries on all discretized surfaces with one boundary made of l_1 spins $+$.

$$\tilde{W}(y) = \sum_{l_2=0}^{\infty} x^{-l_2-1} \sum_{\substack{\text{surfaces,} \\ \text{boundary} = l_2 \text{ down}}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} = $$



$$+ \cdots \tag{2.9}$$

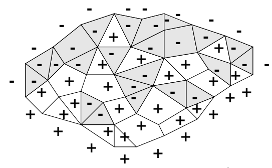
the sum carries on all discretized surfaces with one boundary made of l_2 spins $-$.

We also define:

$$W(x,y) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{1}{y - M_2} \right\rangle = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{T_{k,l}}{x^{k+1} y^{l+1}} \tag{2.10}$$

This function provides information on the correlation between the spectrum of M_1 and that of M_2. Its diagrammatic expansion generates surfaces with one "bi-colored" boundary, i.e. a boundary of lenght $l_1 + l_2$ made of l_1 consecutive spins up, followed by l_2 down:

$$W(x,y) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} x^{-l_1-1}\, y^{-l_2-1} \sum_{\substack{\text{surfaces,} \\ \text{bnd} = l_1 \text{ up} + l_2 \text{ down}}} \sum_{\text{matter}} \mathrm{e}^{-\text{Action}} = $$



$$. \tag{2.11}$$

We also define:

$$U(x,y) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \tag{2.12}$$

$$\tilde{U}(x,y) = \frac{1}{N} \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{1}{y - M_2} \right\rangle \tag{2.13}$$

which are polynomials in one variable and analytical near ∞ in the other variable.

And we define

$$P(x,y) = \frac{1}{N} \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle \tag{2.14}$$

which is a polynomial in both variables, of degree $(d_1 - 1, d_2 - 1)$.

In appendix C we compute another one-loop function $H(x,y,x',y')$, which generates surfaces with one quadri-colored boundary, i.e. surfaces whose boundary is $+-+-$. Somehow it is the random surface analogus of a rectangle with opposite sides of the same spin and adjacent sides of opposite spin. It maybe very usefull to compute that kind of function to study how spin percolates from one side to the other. It may have applications to compute boundary operators in 2d statistical physics on a random surface.

2.2.2 Two-loop functions

We also need to define the following correlation functions, which contain the product of two traces. Their diagrammatic expansion is the generating function for statistical physics models coupled to gravity, on a surface with two boundaries (cylinder-like), this is why they are called two-loop functions.

$$\Omega(x;x') = \frac{\partial W(x)}{\partial V_1(x')} = \left\langle \mathrm{tr}\,\frac{1}{x-M_1}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle - N^2 W(x)W(x')\,, \tag{2.15}$$

$$\tilde\Omega(y;x') = \frac{\partial \tilde W(y)}{\partial V_1(x')} = \left\langle \mathrm{tr}\,\frac{1}{x'-M_1}\,\mathrm{tr}\,\frac{1}{y-M_2}\right\rangle - N^2 W(x')\tilde W(y)\,, \tag{2.16}$$

$$\begin{aligned} U(x,y;x') &= \frac{\partial U(x,y)}{\partial V_1(x')} \\ &= \left\langle \mathrm{tr}\,\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle - N^2 U(x,y)W(x')\,, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \tilde U(x,y;x') &= \frac{\partial \tilde U(x,y)}{\partial V_1(x')} \\ &= \left\langle \mathrm{tr}\,\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{1}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle - N^2 \tilde U(x,y)W(x')\,, \end{aligned} \tag{2.18}$$

$$\begin{aligned} P(x,y;x') &= \frac{\partial P(x,y)}{\partial V_1(x')} \\ &= \left\langle \mathrm{tr}\,\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle - N^2 P(x,y)W(x')\,, \end{aligned} \tag{2.19}$$

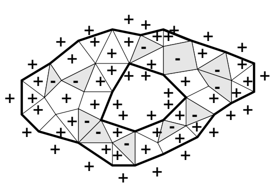
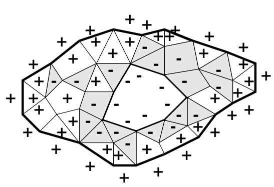
where the "loop-insertion" operators (see [29]) are formally[3] defined as:

$$\frac{\partial}{\partial V_1(x')} = \sum_{k=1}^{\infty}\frac{k}{x'^k}\frac{\partial}{\partial g_k}\,, \qquad \frac{\partial}{\partial V_2(y')} = \sum_{k=1}^{\infty}\frac{k}{y'^k}\frac{\partial}{\partial g_k^*} \tag{2.20}$$

their action on the partition function or on expectation values inserts a trace, i.e. a boundary (a "loop") for the ensemble of surfaces. For instance we have:

$$W(x) = \frac{\partial}{\partial V_1(x)}F\,, \qquad \Omega(x;x') = \frac{\partial}{\partial V_1(x')}W(x) = \frac{\partial}{\partial V_1(x)}\frac{\partial}{\partial V_1(x')}F = \Omega(x';x)\,. \tag{2.21}$$

The diagrammatic expansion of F gives surfaces with no boundary, $W(x)$ gives surfaces with one boundary, $\Omega(x;x')$ gives surfaces with two (spin up) boundaries, and $\tilde\Omega(y;x)$ gives surfaces with one spin up boundary and one spin down boundary. And so on: $\partial/\partial V_1(x')$ inserts a $+$ loop, $\partial/\partial V_2(y')$ inserts a $-$ loop.

$\Omega(x;x') =$  , $\quad \tilde\Omega(y;x') =$  .

[3]By formally, we mean they make sense only through their large x (resp. y) expansion. If k is larger than d_1+1, $\partial/\partial g_k$ means: take a potential of degree $\geq k$, and then take the derivative w.r.t. g_k at $g_j = 0$ for $j > d_1+1$.

3. The loop equations

3.1 Generalities

The loop equations are a consequence of the reparametrization invariance of the matrix integral. Basically, one writes that under an infinitesimal change of variable of the type:

$$M_1 \to \tilde{M}_1 = M_1 + \epsilon f(M_1, M_2) \tag{3.1}$$

the partition function is left unchanged to order 1 in ϵ. We have to compute the variation of the measure and the variation of the potential:

$$\begin{aligned} Z &= \int \mathrm{d}\tilde{M}_1 \mathrm{d}M_2 \, \mathrm{e}^{-N \operatorname{tr}[V_1(\tilde{M}_1)+V_2(M_2)-\tilde{M}_1 M_2]} \\ &= \int \mathrm{d}M_1 \mathrm{d}M_2 \, (1+\epsilon J(M_1,M_2)) \, \mathrm{e}^{-N \operatorname{tr}[V_1(M_1)+V_2(M_2)-M_1 M_2]} (1-\epsilon K(M_1,M_2)) \end{aligned} \tag{3.2}$$

where $J(M_1, M_2)$ is the jacobian of the change of variable computed to order 1 in ϵ:

$$\det \frac{\partial \tilde{M}_1}{\partial M_1} = 1 + \epsilon J(M_1, M_2) + O(\epsilon^2) \tag{3.3}$$

and $K(M_1, M_2)$ is the variation of the action to order 1 in ϵ. We must have:

$$\langle J(M_1, M_2) \rangle = \langle K(M_1, M_2) \rangle \tag{3.4}$$

We are going to give the recipe[4] to compute $J(M_1, M_2)$ for various change of variables $f(M_1, M_2)$.

Remarks:

- It is important to take into account the non-commutativity of matrices.
- It is also important to choose a change of variable which is a *hermitean* matrix.
- We are presenting the method for variations of M_1 but of course the same applies to M_2 and we will use both changes of variables in the body of this article.

3.2 split and merge rules

We will be interested in change of variables of the following form:[5]

- $f(M_1, M_2) = A \frac{1}{x - M_1} B$:

  where the matrices A and B are functions of M_1 and M_2. The variation of the measure is:

  $$J(M_1, M_2) = \operatorname{tr}\left(A \frac{1}{x - M_1}\right) \operatorname{tr}\left(\frac{1}{x - M_1} B\right) + \text{contributions from } A, \text{ and } B\,. \tag{3.5}$$

  somehow, you split the trace into two traces whenever you meet a $\frac{1}{x-M_1}$ term outside a trace, and each side receives a $\frac{1}{x-M_1}$.

[4]Easy to prove directly order by order in x. We leave the proof to the reader. A similar method is used in [35].

[5]As before, $\frac{1}{x-M_1} = \sum_k x^{-k-1} M_1^k$ is a "formal" notation for the collection of M_1^k, $k = 0, \ldots, \infty$.

- $f(M_1,M_2) = A\,\mathrm{tr}\left(B\frac{1}{x-M_1}\right)$:
  where A, B are functions of M_1 and M_2. The variation of the measure is:

$$J(M_1,M_2) = \mathrm{tr}\left(A\frac{1}{x-M_1}B\frac{1}{x-M_1}\right) + \text{contributions from } A, \text{ and } B\,. \tag{3.6}$$

  somehow, you merge the traces whenever you meet a $\frac{1}{x-M_1}$ term inside a trace, and duplicate the $\frac{1}{x-M_1}$ term (one before B, one after B).

- The variations of A and B are computed recursively by the chain rule.

The variation of the action is simply:

$$\frac{1}{N}K(M_1,M_2) = \mathrm{tr}\left(V_1'(M_1)f(M_1,M_2) - M_2 f(M_1,M_2)\right) \tag{3.7}$$

and we will use the following trick very often:

$$\mathrm{tr}\left(V_1'(M_1)\frac{1}{x-M_1}A\right) = V_1'(x)\,\mathrm{tr}\left(\frac{1}{x-M_1}A\right) - \mathrm{tr}\left(\frac{V_1'(x)-V_1'(M_1)}{x-M_1}A\right) \tag{3.8}$$

where it is important to notice that the last term is polynomial in x.

3.3 Factorization "theorem"

Then, we will make the asumption that in the large-N limit, the average of product of traces decouple:

$$\frac{1}{N^2}\left<\mathrm{tr}\,A\,\mathrm{tr}\,B\right> \sim \frac{1}{N}\left<\mathrm{tr}\,A\right>\frac{1}{N}\left<\mathrm{tr}\,B\right> + O\left(\frac{1}{N^2}\right) \tag{3.9}$$

more precisely that the connected correlation function

$$\left<\mathrm{tr}\,A\,\mathrm{tr}\,B\right>_c = \left<\mathrm{tr}\,A\,\mathrm{tr}\,B\right> - \left<\mathrm{tr}\,A\right>\left<\mathrm{tr}\,B\right> \tag{3.10}$$

has a large-N limit.

The factorization "theorem" is not a theorem, it was never proven for the 2-matrix-model,[6] but there are plenty of heuristic reasons to believe that it holds, and it was used by nearly all authors in this field. As we said in the introduction, we are not going to prove the existence of the $1/N^2$ expansion, but assuming it, we are going to give an algorithmic method to compute it.

3.4 The master loop equation

Consider the change of variable:

$$M_2 \to M_2 + \epsilon\frac{1}{x-M_1}$$

[6] It was proven rigorously for the one-matrix model in [31].

JHEP01(2003)051

which is indeed a hermitean matrix. It leads to (there is no variation of the measure since δM_2 is independent of M_2, the r.h.s. is the variation of the potential):

$$0 = \left\langle \mathrm{tr}\, \frac{1}{x-M_1} V_2'(M_2) - \frac{M_1}{x-M_1} \right\rangle \tag{3.11}$$

i.e.

$$\frac{1}{N} \left\langle \mathrm{tr}\, \frac{1}{x-M_1} V_2'(M_2) \right\rangle = xW(x) - 1\,. \tag{3.12}$$

Now, consider the change of variables (which is indeed a hermitean matrix):

$$M_1 \to M_1 + \epsilon \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \frac{1}{x - M_1} + \epsilon \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2}$$

it leads to (the l.h.s. is the variation of the measure, the r.h.s. of the potential, and we use the cyclicity of the trace):

$$\begin{aligned}\left\langle \mathrm{tr}\, \frac{1}{x-M_1}\, \mathrm{tr}\, \frac{1}{x-M_1} \frac{V_2'(y) - V_2'(M_2)}{y-M_2} \right\rangle = N \Bigg\langle \mathrm{tr}\, \frac{V_1'(M_1)}{x-M_1} \frac{V_2'(y) - V_2'(M_2)}{y-M_2} - \\ - \mathrm{tr}\, \frac{1}{x-M_1} M_2 \frac{V_2'(y) - V_2'(M_2)}{y-M_2} \Bigg\rangle \end{aligned} \tag{3.13}$$

i.e. (using eq. (3.12)):

$$(y - Y(x))U(x,y) = V_2'(y)W(x) - P(x,y) - xW(x) + 1 - \frac{1}{N^2} U(x,y;x)\,. \tag{3.14}$$

Notice that $U(x,y)$ is a polynomial in y, therefore it is finite for $y = Y(x)$. If we choose $y = Y(x)$ we get (using $W(x) = V_1'(x) - Y(x)$):

$$\begin{cases} y = Y(x) \\ (V_2'(y) - x)(V_1'(x) - y) - P(x,y) + 1 = \frac{1}{N^2} U(x,y;x) \end{cases} \tag{3.15}$$

which we write:

$$\boxed{E(x,Y(x)) = \frac{1}{N^2} U(x,Y(x);x)} \tag{3.16}$$

where $E(x,y)$ is a polynomial in x (degree d_1+1) and y (degree d_2+1):

$$\boxed{E(x,y) = (V_1'(x) - y)(V_2'(y) - x) - P(x,y) + 1} \tag{3.17}$$

eq. (3.16) is called the "master loop equation" [36], we will see below that it allows to determine the function $Y(x)$ and all other functions in the problem.

3.5 $1/N^2$ expansion

We assume that the function $Y(x)$ can be written as a power series in $1/N^2$:

$$Y(x) = Y^{(0)}(x) + \frac{1}{N^2} Y^{(1)}(x) + \cdots \tag{3.18}$$

Such an expansion is believed to hold in the "1-cut case", which will be explained below (section 4.1).

- To leading order, the loop equation is an algebraic equation:

$$E^{(0)}(x, Y^{(0)}(x)) = 0 \tag{3.19}$$

where

$$E^{(0)}(x,y) = (V_1'(x) - y)(V_2'(y) - x) - P^{(0)}(x,y) + 1\,. \tag{3.20}$$

The polynomial $P^{(0)}(x,y)$ has degrees $(d_1 - 1, d_2 - 1)$, it is determined by the large x and large y behaviors of $Y(x)$ and $X(y)$, as well as some analytical requirement, namely the "one-cut asumption". As we will see below, in the 2-matrix model, this asumption is replaced by the asumption that the algebraic curve $E(x,y) = 0$ has genus zero. In the next section we will study this algebraic curve, and we will explain how to compute the polynomial $P^{(0)}(x,y)$.

Let us note that this algebraic equation was derived by many authors and various methods in the literature: with the the loop equation method, the existence of an algebraic equation is proven (and computed for small degree potentials) in [36], and was presented for arbitrary polynomial potentials in the appendix of [37] and [38]. The authors of [39] and [27] derived it by the "saddle point method" inspired from [40], and from the 2-toda hierarchy dispersionless limit, and that algebraic equation was used by many other authors in particular cases. And notice that it is the same equation as the spectral curve of any of the 4 dual differential systems in [23].

- To order $1/N^2$ we expand:

$$E(x,y) = E^{(0)}(x,y) - \frac{1}{N^2} P^{(1)}(x,y) + \cdots \tag{3.21}$$

where $P^{(1)}(x,y)$ has degrees $(d_1 - 1, d_2 - 1)$. We will see how to compute it in section 6. We easily get:

$$Y^{(1)}(x) = \frac{U^{(0)}(x, Y^{(0)}(x); x) + P^{(1)}(x, Y^{(0)}(x))}{\partial_y E^{(0)}(x, Y^{(0)}(x))}\,. \tag{3.22}$$

In this expression, all terms in the r.h.s. are computed to leading order, except the polynomial $P^{(1)}(x,y)$. But $P^{(1)}(x,y)$ is completely determined (in terms of leading order functions only) by the condition that all the unwanted zeroes of the denominator (those which don't correspond branch-points) cancel. In other words, once we know the functions $Y(x)$ and $U(x,y;x')$ to leading order, we can compute $Y(x)$ to order 1. Similarly, we can compute every loop function to order 1.

- We can repeat the procedure to find higher orders. Once we know all the loop functions up to order h, we can compute them to order $h{+}1$. Of course the expressions become more and more complicated with increasing orders, but the method is the same. We will illustrate our method by computing the order 1 only.

- once we know $Y^{(h)}(x)$, we can find the order h free energy $F^{(h)}$, by integration: $Y^{(h)}(x) = -\partial F^{(h)} / \partial V_1(x)$.

JHEP01(2003)051

4. Leading order: algebraic equation

While we study the leading order, we will drop the subscripts (0) for lisibility.

4.1 Reminder: one-cut asumption for the 1-matrix model

In the 1-matrix model, the existence of the $1/N^2$ topological expansion is related to the 1-cut asumption. We refer the reader to [41, 42, 31] for more detailed explanations, and we just summarize the main idea.

When there is more than one cut, i.e. when the density of eigenvalues has a disconnected support, there are tunnel effects between the different cuts which generate an oscillating function of N (a theta-function). A periodic function cannot have a $1/N^2$ expansion. The frequencies of oscillations are the filling fraction of eigenvalues in the cuts, in other words, the frequencies are equal to the integrals of the 1-loop function $W(x)$ along non-trivial cycles of the hyper-elliptical surface generated by $(x, W(x))$ (In the 1-matrix case the surface is hyperelliptical, because $W(x)$ obeys a quadratic equation). It is only in the one-cut case that there is no oscillations (there is no non-trivial cycle), and the $1/N^2$ expansion can exist.

We expect the same idea to hold for the 2-matrix model. The method of [41] can be adapted to the 2-matrix case: the frequencies are the contour integrals of $W(x)dx$, which are the same as the contour integrals of $Y(x)dx$ (which are also the same as the integrals of $X(y)dy$ by integration by parts), along the non-trivial cycles of the algebraic curve $(x, Y(x))$. The $1/N^2$ expansion cannot exist if there are oscillations. Therefore we will require that there is no oscillation, i.e. that there is no non-trivial cycle, i.e. that the algebraic curve has genus zero.

Notice a difference between the 1-matrix case and the 2-matrix case:

The number of connected parts of the support of the density is actually the number of cuts in the first sheet only. There may be cuts in other sheets.

- In the 1-matrix case, the curve is hyper-elliptical, it has only two sheets, thus the first sheet contains all the cuts. The number of non-trivial cycles is equal to the number of cuts (in the first sheet) minus one. Therefore in the 1-matrix model, the genus zero asumption is equivalent to the 1-cut asumption and is equivalent to a density with connected support.

- In the 2-matrix case, the curve is only algebraic. It can have more than two sheets, so that the non-trivial cycles are not necessarily in the first sheet. One cut in the first sheet is not equivalent to genus zero. In fact, there are $d_2 + 1$ sheets and at least d_2 cuts, at least $d_2 - 1$ of them are NOT in the first sheet. In the 2-matrix case, the connectedness of the support of the density is a necessary condition, but is not sufficient. The necessary and sufficient condition is the genus zero condition.

Therefore, in the 2-matrix model, we claim that the existence of the $1/N^2$ expansion is related to the asumption that the algebraic curve $E^{(0)}(x, y) = 0$ has genus zero.

From now on, we will consider only the genus zero case. We would like to emphasize that many of the results derived below for the leading order loop functions, can be derived

JHEP01(2003)051

for arbitrary genus with no difficulty (for instance the functions Ω are always the Bargmann kernels), but are out of the scope of this article. Many of them are presented in [24].

The previous discussion applies to the case where the partition function eq. (2.1) is well defined, i.e. the potentials satisfy condition eq. (2.3). However, there is another case of "existence" of the $1/N^2$ expansion: when the genus h free energy $F^{(h)}$ is formally defined by its diagrammatic expansion eq. (1.4) and eq. (2.4). The formal series for $F^{(h)}$ is obtained by expanding in the vicinity of a gaussian integral [3, 10], i.e. by considering that the potentials are "small deformations" of quadratic potentials. When both potentials are quadratic, the algebraic curve $E(x, y) = 0$ clearly has genus zero. The formal expansion in power series of the g_k's and g_l^*'s can not change the genus. Therefore, the formal diagrammatic expansion of $F^{(h)}$ is also obtained by a genus zero assumption.

4.2 Genus zero asumption

Since we require that $E(x, y) = 0$ be a genus zero algebraic curve, there must exist a rational uniformization. Such a rational uniformization appears naturally in the framework of bi-orthogonal polynomials as shown in [23, 37, 38, 19]. That leads us to choose it of the following form:

$$x = \mathcal{X}(s) = \gamma s + \sum_{k=0}^{d_2} \frac{\alpha_k}{s^k}\,, \qquad y = \mathcal{Y}(s) = \frac{\gamma}{s} + \sum_{j=0}^{d_1} \beta_j s^j\,. \tag{4.1}$$

This means that for every (x, y) which satisfy $E(x, y) = 0$, there exists at least one s such that $x = \mathcal{X}(s)$ and $y = \mathcal{Y}(s)$.

We note the inverse functions:

$$x = \mathcal{X}(s) \leftrightarrow s = \sigma(x)\,, \qquad y = \mathcal{Y}(s) \leftrightarrow s = \tilde{\sigma}(y)\,. \tag{4.2}$$

The functions $\sigma(x)$ and $\tilde{\sigma}(y)$ are multivalued, we will discuss their sheet structure below. The functions $Y(x)$ and $X(y)$ are:

$$Y(x) = \mathcal{Y}(\sigma(x))\,, \qquad X(y) = \mathcal{X}(\tilde{\sigma}(y)) \tag{4.3}$$

They are multivalued too, and their sheet structure will be discussed below.

4.3 The parameters α_k and β_j

The α's and β's are merely a reparametrization of the coefficients of the potentials.

Indeed, since for large x we have: $V_1'(x) - Y(x) = W(x) \sim \frac{1}{x} + O(1/x^2)$ (this corresponds to $s \to \infty$), and for large y we have: $V_2'(y) - X(y) = \tilde{W}(y) \sim \frac{1}{y} + O(1/y^2)$ $(s \to 0)$, we must have:

$$\begin{aligned} \mathcal{Y}(s) - V_1'(\mathcal{X}(s)) &\underset{s\to\infty}{\sim} -\frac{1}{\gamma s} + O(1/s^2) \\ \mathcal{X}(s) - V_2'(\mathcal{Y}(s)) &\underset{s\to 0}{\sim} -\frac{s}{\gamma} + O(s^2) \end{aligned} \tag{4.4}$$

Both these equations imply that (compute $\text{Res}\mathcal{Y}(s)\mathcal{X}'(s)\text{d}s = -\text{Res}\mathcal{X}(s)\mathcal{Y}'(s)\text{d}s$):

$$\boxed{\gamma^2 = -1 + \sum_{k=1}^{\min(d_1,d_2)} k\alpha_k\beta_k\,.} \tag{4.5}$$

And eqs. (4.4) allow to write the coupling constants g_l and g_l^* as functions of the α_k's and β_k's (compute $\text{Res}V_1'(x)x^{l-1}\text{d}x$ and $\text{Res}V_2'(y)y^{l-1}\text{d}y$):

$$g_{l+1} = -\frac{1}{\gamma^{l+1}}\sum_{k=-1}^{d_1-l-1}\left(\prod_{p=0}^{d_2}\sum_{i_p=0}^{[d_1/(p+1)]}\right)k\frac{(l+\sum_p i_p)!}{l!\prod_p i_p!}\,\alpha_k\beta_{(l+k+1+\sum_p(p+1)i_p)}\prod_{p=0}^{d_2}\left(-\frac{\alpha_p}{\gamma}\right)^{i_p} \tag{4.6}$$

$$g^*_{l+1} = -\frac{1}{\gamma^{l+1}}\sum_{k=-1}^{d_2-l-1}\left(\prod_{p=0}^{d_1}\sum_{i_p=0}^{[d_2/(p+1)]}\right)k\frac{(l+\sum_p i_p)!}{l!\prod_p i_p!}\,\beta_k\alpha_{(l+k+1+\sum_p(p+1)i_p)}\prod_{p=0}^{d_1}\left(-\frac{\beta_p}{\gamma}\right)^{i_p} \tag{4.7}$$

where we take the convention that $\beta_j = 0$ if $j > d_1$ and $\alpha_j = 0$ if $j > d_2$, and $\alpha_{-1} = \beta_{-1} = \gamma$.

As an example, for $l = d_1$ and $l = d_1 - 1$, the formula reduces to:

$$g_{d_1+1} = \frac{\beta_{d_1}}{\gamma^{d_1}}\,, \qquad g_{d_1} = \frac{\beta_{d_1-1}}{\gamma^{d_1-1}} - d_1\alpha_0\frac{\beta_{d_1}}{\gamma^{d_1}}\,, \qquad \ldots \tag{4.8}$$

In principle, it should be possible to revert these formula, and compute the α's and β's as functions of the coupling constants. This can be done at least numerically.

Remark: notice that there might exist more than one solution. Only the solution which leads to an absolute minimum of the leading order free energy $F^{(0)}$ should be considered. If the potentials V_1 and V_2 are real (which is the case for most physical models), the density of eigenvalues of the first (resp. second) matrix is the imaginary part of $Y(x)$ (resp. $X(y)$) along the cut, and the density must be positive. Any solution for the α's and β's which does not satisfy that is unacceptable. If there is no acceptable solution, this means that our choice of V_1 and V_2 does not correspond to a genus zero case. In general, if V_1 and V_2 have only one well, or if they are close to quadratic potentials, we have a genus zero solution.

4.4 Sheet structure of the function $Y(x)$

The function $\sigma(x)$ is multivalued, indeed the equation

$$\mathcal{X}(s) = x \tag{4.9}$$

has $d_2 + 1$ solutions, which we note

$$\{\sigma_0(x), \sigma_1(x), \ldots, \sigma_{d_2}(x)\} \tag{4.10}$$

and therefore, the function $Y(x) = \mathcal{Y}(\sigma(x))$ is multivalued with $d_2 + 1$ values which we note:

$$\{Y_0(x), Y_1(x), \ldots, Y_{d_2}(x)\} \quad \text{where } Y_k(x) = \mathcal{Y}(\sigma_k(x))\,. \tag{4.11}$$

We define $\sigma(x) = \sigma_0(x)$ to be the "physical sheet" solution, i.e. the one such that $\sigma(x) \to \infty$ when $x \to \infty$, i.e. the one for which $W(x) = V_1'(x) - Y(x) \sim O(1/x)$. The other solutions $\sigma_k(x)$ with $k = 1, \ldots, d_2$ all tend to zero when $x \to \infty$.

4.4.1 Endpoints and cuts

The functions $Y(x)$ and $\sigma(x)$ have cuts, whith a generic square root behaviour near the endpoints, which means that the derivative with respect to x diverges at the endpoints. Since $Y'(x) = \mathcal{Y}'(s)/\mathcal{X}'(s)$, the endpoints are solutions of $s=0$ or solutions of the equation:

$$\mathcal{X}'(s) = 0 \tag{4.12}$$

which has d_2+1 solutions. We note them:

$$\{e_0, e_1, \dots, e_{d_2}\} \tag{4.13}$$

the endpoints are thus located at the $\mathcal{X}(e_k)$'s, as well as at infinity. In the physical sheet, the function $Y(x)$ is analytical at infinity, and the only cut in the physical sheet is a line joining two endpoints of the type $\mathcal{X}(e_k)$. In the other sheets, the function $Y(x)$ behaves as x^{1/d_2} near infinity, the cuts are lines joining some $\mathcal{X}(e_k)$ to ∞. There are exactly d_2 cuts, in agreement with the fact that the genus is zero.

Remark: there is some arbitrariness in the choice of cuts. The cuts can be any curves which link the endpoints. If the potentials are real, one usually chooses the cut in the physical sheet as a segment of the real axis. A possible canonical definition of the cuts is motivated by the large-N asymptotics of the associated bi-orthogonal polynomials and the Riemann-Hilbert problem [37, 38, 25, 24], and can be stated as follows:

$$\left\{ x \ / \ \exists i \neq j \,,\ \mathrm{Re} \int_{\sigma_i(x)}^{\sigma_j(x)} \mathcal{Y}(s)\mathcal{X}'(s)\mathrm{d}s = 0 \right\} . \tag{4.14}$$

Similarly, the function $X(y)$ has exactly d_1 cuts, whose endpoints are $\mathcal{Y}(\tilde{e}_j)$'s and ∞, where

$$\{\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_{d_1}\} \tag{4.15}$$

are the d_1+1 roots of

$$\mathcal{Y}'(s) = 0 \,. \tag{4.16}$$

The cut in the physical sheet is a line joining two endpoints of the type $\mathcal{Y}(\tilde{e}_k)$, and the cuts in the other sheets are lines joining some $\mathcal{Y}(\tilde{e}_k)$ to ∞.

As said in the introduction, we assume that the potentials are not critical, which means that the algebraic curve has no singular points (other than $s=0$ and $s=\infty$). That means that all the e_k and $\tilde{e}_l$ must be distinct (examples of critical potentials are given in [19]: those which produce the rational minimal conformal models (p,q)). The non-criticality asumption, is crucial for the derivations we are going to present below: we will often have $\mathcal{X}''(e_k)$ or $\mathcal{Y}'(e_k)$ in denominators.

4.4.2 Sheets in the s-plane

We define the x-sheets (resp. y-sheets) in the s-plane, as domains which contain only one $\sigma_k(x)$ (resp. $\tilde{\sigma}_k(y)$) when x (resp. y) sweeps the complex plane. In other words, the function $\mathcal{X}(s)$ (resp. $\mathcal{Y}(s)$) restricted to one domain is one to one.

The domains are separated by closed curves in the s-plane, which are the projections of the cuts in the $x-plane$ (resp. y-plane).

The x-sheets (resp. y-sheets) have the following properties:

- The physical sheet is the sheet which contains ∞ (resp. 0).
- All the d_2 (resp. d_1) other sheets contain 0 (resp. ∞).

And the cuts:

- The cut in the physical sheet corresponds to a contour in the s-plane which encircles ∞ (resp. 0). If the potentials are real, we expect the cut in the x-plane (reps. y-plane) to be on the real axis, and its s-plane image to be a contour which crosses the real axis, and symmetric with respect to the real axis. The two endpoints are real solutions of $\mathcal{X}'(s) = 0$ (resp. $\mathcal{Y}'(s) = 0$).
- The other $d_2 - 1$ (reps. $d_1 - 1$) cuts are contours in the other sheets. They all meet in 0 (resp. ∞). One of their endpoint is a solution of $\mathcal{X}'(s) = 0$ (resp. $\mathcal{Y}(s) = 0$) and the other endpoint is at $s = 0$ (reps. $s = \infty$).

Notational remark: when we write $\sigma(x)$ this means that $\sigma(x)$ is considered as a multivalued function. For instance the multivalued function $Y(x)$ is equal to

$$Y(x) = \mathcal{Y}(\sigma(x)) \,. \tag{4.17}$$

We write $\sigma_k(x)$ or $Y_k(x)$ only when we want to specify in which sheet is x. By default, we will consider that x is in the physical sheet, and therefore $\sigma_0(x)$ is in the domain which contains $s = \infty$, and $\sigma_k(x)$ with $k \geq 1$ is in a domain which contains $s = 0$. Note that any symmetric function of $(\sigma_0(x), \dots, \sigma_{d_2}(x))$ is a regular function of x, i.e. it has no cut.

4.5 The polynomial $P(x, y)$

The polynomial $P(x, y)$ is found from $E(\mathcal{X}(s), \mathcal{Y}(s)) = 0$, i.e.:

$$P(\mathcal{X}(s), \mathcal{Y}(s)) = (V_1'(\mathcal{X}(s)) - \mathcal{Y}(s))(V_2'(\mathcal{Y}(s)) - \mathcal{X}(s)) + 1 \tag{4.18}$$

both sides are Laurent polynomials of s of the same degree. By indentifying the coefficients of powers of s on both sides, one can compute explicitely all the coefficients of the polynomial $P(x, y)$. The expression of $P(x, y)$ in terms of α_k and β_j can be written explicitely with the determinant formula:

$$E(x,y) = \frac{1}{\gamma^{d_1+d_2}} \det \begin{pmatrix} \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{d_2} & 0 & \dots & 0 \\ 0 & \ddots & & & & \ddots & & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & \dots & 0 & \gamma & \alpha_0 - x & \alpha_1 & \dots & \alpha_{d_2} \\ \beta_{d_1} & \dots & \beta_1 & \beta_0 - y & \gamma & 0 & \dots & 0 \\ 0 & \ddots & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & \dots & 0 & \beta_{d_1} & \dots & \beta_1 & \beta_0 - y & \gamma \end{pmatrix} \tag{4.19}$$

JHEP01(2003)051

This determinant of size d_1+d_2+2 is the discriminant which vanishes if and only if the Laurent polynomials in s: $\mathcal{X}(s)-x$ and $\mathcal{Y}(s)-y$ have a common root.

There is also the formula:

$$E(x,y) = -(g_{d_1+1})^{d_2+1}\,\frac{1}{\gamma^{2d_2}}\prod_{i=0}^{d_2}\prod_{j=0}^{d_1}(\gamma\sigma_i(x)-\gamma\tilde\sigma_j(y)) \tag{4.20}$$

indeed, it is a symmetric function of the $\sigma_i(x)$ and $\tilde\sigma_j(y)$, with no poles, therefore it is a polynomial in x and y, and it vanishes when $\mathcal{X}(s)-x$ and $\mathcal{Y}(s)-y$ have a common root.

Remark: had we not made the genus zero asumption, we could determine $P(x,y)$ by a determinant formula[7] very similar to eq. (4.19). We can also determine $P(x,y)$ by requiring that $E(x,y)=0$ is a genus g curve, and that all the b-cycle integrals of $Y(x)\mathrm{d}x$ vanish. There is one (or more) solution for all g. For any g between 0 and $\left[\frac{d_1 d_2-1}{2}\right]$ we can compute all the loop functions (to leading order), and therefore we can compute the free energy $F^{(0)}(g)$ as a function of the genus g. The actual value of g is the one which minimizes $F^{(0)}$. The genus zero asumption is thus that the minimum is obtained for $g=0$.

5. Leading order loop functions

5.1 Some leading order loop equations

The change of variables (we drop the $O(1/N^2)$ terms):

- $M_1 \to M_1 + \epsilon\frac{1}{x-M_1}\frac{1}{y-M_2} + \epsilon\frac{1}{y-M_2}\frac{1}{x-M_1}$ implies:
$$(y-Y(x))W(x,y) = W(x)-\tilde U(x,y) = V_1'(x)-Y(x)-\tilde U(x,y)\,; \tag{5.1}$$
- $M_2 \to M_2 + \epsilon\frac{1}{x-M_1}\frac{1}{y-M_2} + \epsilon\frac{1}{y-M_2}\frac{1}{x-M_1}$ implies:
$$(x-X(y))W(x,y) = \tilde W(y)-U(x,y) = V_2'(y)-X(y)-U(x,y)\,; \tag{5.2}$$
- $M_1 \to M_1 + \epsilon\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2} + \epsilon\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\frac{1}{x-M_1}$ implies:
$$(y-Y(x))U(x,y) = -P(x,y)+V_2'(y)W(x)-\frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{x-M_1}V_2'(M_2)\right\rangle; \tag{5.3}$$
- $M_2 \to M_2 + \epsilon\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{1}{y-M_2} + \epsilon\frac{1}{y-M_2}\frac{V_1'(x)-V_1'(M_1)}{x-M_1}$ implies:
$$(x-X(y))\tilde U(x,y) = -P(x,y)+V_1'(x)\tilde W(y)-\frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{y-M_2}V_1'(M_1)\right\rangle; \tag{5.4}$$
- $M_2 \to M_2 + \epsilon\frac{1}{x-M_1}$ implies:
$$\frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{x-M_1}V_2'(M_2)\right\rangle = xW(x)-1\,; \tag{5.5}$$
- $M_1 \to M_1 + \epsilon\frac{1}{y-M_2}$ implies:
$$\frac{1}{N}\left\langle \mathrm{tr}\,\frac{1}{y-M_2}V_1'(M_1)\right\rangle = y\tilde W(y)-1\,; \tag{5.6}$$

[7] Thanks to J. Hurtubise for that remark.

5.2 The function $U(x,y)$

Using eq. (5.3) and eq. (5.5), we find that to leading order, the function $U(x,y)$ is simply:

$$\boxed{U(x,y) = V_2'(y) - x + \frac{E(x,y)}{y - Y(x)}}. \tag{5.7}$$

Notice that this function has no pole, and it is indeed a polynomial in y.

5.3 $W(x,y)$

Using eq. (5.2) and eq. (5.7), we find that, to leading order, the function $W(x,y)$ is:

$$\boxed{W(x,y) = 1 - \frac{E(x,y)}{(x - X(y))(y - Y(x))}} \tag{5.8}$$

It can also be written:

$$W(x,y) = 1 - \left(\frac{\beta_{d_1}}{\gamma}\right)^{d_2} \frac{\sigma_0(x)\tilde{\sigma}_0(y)^{d_2}}{\sigma_0(x) - \tilde{\sigma}_0(y)} \prod_{i=1}^{d_2}\prod_{j=1}^{d_1}(\sigma_i(x) - \tilde{\sigma}_j(y)) \tag{5.9}$$

it has a pole when $\sigma_0(x) = \tilde{\sigma}_0(y)$.

We are now going to compute the two-loop functions.

5.4 The functions $\Omega(x;x')$ and $\tilde{\Omega}(y;x')$

These functions were first computed by [19] using the orthogonal polynomial's technics, they can also be found by another method, namely: they must be rational functions of $\sigma(x)$ (resp. $\tilde{\sigma}(y)$) and $\sigma(x')$, and they are determined completely by their poles and behaviours near ∞. The result is the following:

$$\begin{aligned}
\Omega(x;x') &= \frac{\partial}{\partial V_1(x')} W(x) = \partial_x \partial_{x'} \ln \frac{\sigma(x) - \sigma(x')}{x - x'} = -\frac{1}{(x-x')^2} + \frac{\sigma'(x)\sigma'(x')}{(\sigma(x) - \sigma(x'))^2} \\
&= -\frac{1}{(x-x')^2} + \frac{1}{\mathcal{X}'(\sigma(x))\mathcal{X}'(\sigma(x'))(\sigma(x) - \sigma(x'))^2}
\end{aligned} \tag{5.10}$$

and since $Y(x) = V_1'(x) - W(x)$ we have:

$$\frac{\partial}{\partial V_1(x')} Y(x) = -\frac{\sigma'(x)\sigma'(x')}{(\sigma(x) - \sigma(x'))^2} = -\partial_x \partial_{x'} \ln(\sigma(x) - \sigma(x')) \tag{5.11}$$

i.e. $\frac{\partial}{\partial V_1(x')} Y(x)$ is the Bargmann kernel.[8]

At $x = x'$ we get the Schwarzian derivative of $\sigma(x)$:

$$\Omega(x;x) = \frac{1}{6}\frac{\sigma'''(x)}{\sigma'(x)} - \frac{1}{4}\frac{\sigma''(x)^2}{\sigma'(x)^2}\,. \tag{5.12}$$

[8]Without the genus zero asumption, one has $\partial Y^{(0)}(x)/\partial V_1(x') =$ Bargman kernel, but $\partial Y(x)/\partial V_1(x') =$ square of the Szegö kernel $+O(1/N)$.

Similarly [19], the function $\tilde{\Omega}(y;x')$ is:

$$\tilde{\Omega}(y;x') = \frac{\partial}{\partial V_1(x')}\tilde{W}(y) = -\partial x'\partial_y \ln\left(\sigma(x') - \tilde{\sigma}(y)\right) = -\frac{\sigma'(x')\tilde{\sigma}'(y)}{(\sigma(x') - \tilde{\sigma}(y))^2} \tag{5.13}$$

i.e.:

$$\frac{\partial}{\partial V_1(x')}X(y) = \frac{\tilde{\sigma}'(y)\sigma'(x')}{(\tilde{\sigma}(y) - \sigma(x'))^2} = \partial_y\partial_{x'}\ln\left(\tilde{\sigma}(y) - \sigma(x')\right). \tag{5.14}$$

5.5 More loop equations

- The change of variables $\delta M_2 = \frac{1}{x-M_1}\,\mathrm{tr}\,\frac{1}{x'-M_1}$ implies:

$$\left\langle \mathrm{tr}\,\frac{1}{x-M_1}V_2'(M_2)\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle_{\mathrm{c}} = x\Omega(x;x'). \tag{5.15}$$

- Using this result, the change of variables $\delta M_1 = \frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1} + \mathrm{h.c.}$ implies:

$$\begin{aligned}(y - Y(x))U(x,y;x') &= (V_2'(y) - x - U(x,y))\Omega(x;x') - P(x,y;x') - \\ &\quad -\partial_{x'}\frac{U(x,y) - U(x',y)}{x - x'}\end{aligned} \tag{5.16}$$

i.e., using eq. (5.7):

$$(y - Y(x))U(x,y;x') = -\Omega(x;x')\frac{E(x,y)}{y - Y(x)} - P(x,y;x') - \partial_{x'}\frac{U(x,y) - U(x',y)}{x - x'} \tag{5.17}$$

which can also be written, using eq. (5.7) again:

$$\begin{aligned}(y - Y(x))U(x,y;x') &= -(\Omega(x;x') + \frac{1}{(x-x')^2})\frac{E(x,y)}{y - Y(x)} - P(x,y;x') + \\ &\quad +\partial_{x'}\frac{E(x',y)}{(x - x')(y - Y(x'))}.\end{aligned} \tag{5.18}$$

- The change of variables $\delta M_1 = \frac{1}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}$ implies:

$$\left\langle \mathrm{tr}\,V_1'(M_1)\frac{1}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1}\right\rangle_{\mathrm{c}} = y\tilde{\Omega}(y;x') - \partial_{x'}W(x',y). \tag{5.19}$$

- Using this result, the change of variables $\delta M_2 = \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{1}{y-M_2}\,\mathrm{tr}\,\frac{1}{x'-M_1} + \mathrm{h.c.}$ implies:

$$(x - X(y))\tilde{U}(x,y;x') = \tilde{\Omega}(y;x')(V_1'(x) - \tilde{U}(x,y) - y) - P(x,y;x') + \partial_{x'}W(x',y) \tag{5.20}$$

which can also be written:

$$(x - X(y))\tilde{U}(x,y;x') = -P(x,y;x') - \tilde{\Omega}(y;x')\frac{E(x,y)}{x - X(y)} + \partial_{x'}W(x',y). \tag{5.21}$$

5.6 Determination of the polynomial $P(x,y;x')$

In particular, if for any $k=0,\dots,d_2$, we choose $y=Y_k(x)$ in eq. (5.21), we have $X(y)=x$, and using eq. (5.8):

$$P(x,Y_k(x);x') = \partial_{x'}\left[\frac{E(x',Y_k(x))}{(x-x')(Y_k(x)-Y(x'))}\right] - \tilde\Omega(Y_k(x);x')E_x(x,Y_k(x)) \tag{5.22}$$

i.e.

$$P(x,Y_k(x);x') = \partial_{x'}\left[\frac{E(x',Y_k(x))}{(x-x')(Y_k(x)-Y(x'))} + \frac{E_x(x,Y_k(x))}{\mathcal{Y}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))}\right]. \tag{5.23}$$

We know that P is a polynomial in y of degree d_2-1, and we know its value in d_2+1 points, therefore we can determine it with the interpolation formula (see appendix A), which, in this case, reduces to:

$$P(x,y;x') = \sum_{k=0}^{d_2} \frac{P(x,Y_k(x);x')}{(y-Y_k(x))}\,\frac{E(x,y)}{E_y(x,Y_k(x))}\,. \tag{5.24}$$

The latter expression seems to be a polynomial of degree d_2 in y, but one can check that the leading term vanishes (see appendix B). Thus we may also write:

$$P(x,y;x') = \sum_{k=1}^{d_2} \frac{P(x,Y_k(x);x')(Y_k(x)-Y_0(x))}{(y-Y_k(x))(y-Y_0(x))}\,\frac{E(x,y)}{E_y(x,Y_k(x))}\,. \tag{5.25}$$

Both expressions give either:

$$\begin{aligned} P(x,y;x') =& \partial_{x'}\sum_{k=1}^{d_2} \frac{E_x(x,Y_k(x))E(x,y)(Y_k(x)-Y_0(x))}{\mathcal{Y}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))E_y(x,Y_k(x))(y-Y_k(x))(y-Y_0(x))} + \\ &+\frac{E(x',Y_k(x))}{(x-x')(Y_k(x)-Y(x'))}\,\frac{E(x,y)}{E_y(x,Y_k(x))}\,\frac{Y_k(x)-Y_0(x)}{(y-Y_k(x))(y-Y_0(x))}\,. \end{aligned} \tag{5.26}$$

Using $E(\mathcal{X}(s),\mathcal{Y}(s))=0$ we have $\mathcal{Y}'(\sigma_k(x))E_y(x,Y_k(x)) = -\mathcal{X}'(\sigma_k(x))E_x(x,Y_k(x))$, and we can write:

$$\begin{aligned} P(x,y;x') \;=\; & \partial_{x'}\sum_{k=1}^{d_2} \frac{E(x,y)}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))}\,\frac{Y_0(x)-Y_k(x)}{(y-Y_k(x))(y-Y_0(x))} + \\ &+\frac{E(x',Y_k(x))}{(x-x')(Y_k(x)-Y(x'))}\,\frac{E(x,y)}{E_y(x,Y_k(x))}\,\frac{Y_k(x)-Y_0(x)}{(y-Y_k(x))(y-Y_0(x))} \end{aligned} \tag{5.27}$$

or, using eq. (5.24):

$$\begin{aligned} -P(x,y;x') \;=\; & \partial_{x'}\textstyle\sum_{k=0}^{d_2}\displaystyle \frac{E(x,y)}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))}\,\frac{1}{(y-Y_k(x))} - \\ &-\frac{E(x',Y_k(x))}{(x-x')(Y_k(x)-Y(x'))}\,\frac{E(x,y)}{E_y(x,Y_k(x))}\,\frac{1}{(y-Y_k(x))}\,. \end{aligned} \tag{5.28}$$

Eq. (5.28) is clearly a polynomial in y, let us explain how one can check that it is also a polynomial in x: it is symmetric in the $\sigma_k(x)$'s, therefore it has no cut (it takes the same value in all sheets), it must be a rational function of x. The only possible poles could be at $x=x'$, $\mathcal{X}'(\sigma_k(x))=0$ or $E_y(x,Y_k(x))=0$.

- Take $x = x'$, more precisely, for some $k \in [0, d_2]$, $\sigma_k(x) = \sigma(x')$, because of the symmetry one may assume that $k = 0$, and one sees that expression eq. (5.27) has no pole at $\sigma_0(x) = \sigma(x')$.

- Now choose some k and x such that $\mathcal{X}'(\sigma_k(x)) = \epsilon$ is very small. There must exist some (unique because our potentials are non critical) $j \neq k$ such that $\sigma_j(x) - \sigma_k(x) = O(\epsilon)$, and one has $\mathcal{X}'(\sigma_j(x)) = -\mathcal{X}'(\sigma_k(x)) + O(\epsilon^2)$. The poles for j and k thus cancel each other.

- Now choose some k and x such that $E_y(x, Y_k(x)) = \epsilon$ is very small. This means there must exist some (unique because our potentials are non critical) $j \neq k$ such that $Y_j(x) - Y_k(x) = O(\epsilon)$, and one has $E_y(x, Y_j(x)) = -E_y(x, Y_k(x)) + O(\epsilon^2)$. The poles for j and k thus cancel each other in eq. (5.28).

Thus eq. (5.28) has no pole, therefore it is a polynomial.

5.7 Determination of $U(x, y; x')$

We can now compute $U(x, y; x')$ using eq. (5.18):

$$\begin{aligned}(y - Y_0(x))U(x, y; x') = {} & \partial_{x'} \sum_{k=1}^{d_2} \frac{E(x, y)}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x) - \sigma(x'))(y - Y_k(x))} - \\ & - \frac{E(x, y)}{(x - x')} \sum_{k=0}^{d_2} \frac{E(x', Y_k(x))}{(Y_k(x) - Y(x'))E_y(x, Y_k(x))(y - Y_k(x))} + \\ & + \frac{E(x', y)}{(x - x')(y - Y(x'))}\end{aligned} \tag{5.29}$$

which simplifies to:

$$\begin{aligned}(y - Y_0(x))U(x, y; x') = {} & \partial_{x'} \sum_{k=1}^{d_2} \frac{E(x, y)}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x) - \sigma(x'))(y - Y_k(x))} + \\ & + \frac{E(x, y)}{(x - x')(y - Y(x'))} \sum_{k=0}^{d_2} \frac{E(x', Y_k(x))}{E_y(x, Y_k(x))(Y_k(x) - y)} + \\ & + \frac{E(x', y) - E(x, y)}{(x - x')(y - Y(x'))} .\end{aligned} \tag{5.30}$$

Note the following identity (see appendix B, Lemma B.5):

$$\sum_{k=0}^{d_2} \frac{\prod_{j=0}^{d_2}(Y_k(x) - Y_j(x'))}{(Y_k(x) - y)\prod_{j\neq k}(Y_k(x) - Y_j(x))} = 1 - \frac{\prod_j (y - Y_j(x'))}{\prod_j (y - Y_j(x))} \tag{5.31}$$

which implies:

$$\sum_{k=0}^{d_2} \frac{E(x', Y_k(x))}{(Y_k(x) - y)E_y(x, Y_k(x))} = 1 - \frac{E(x', y)}{E(x, y)} \tag{5.32}$$

and thus:

$$\boxed{U(x,y;x') = \partial_{x'} \sum_{k=1}^{d_2} \frac{E(x,y)}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))(y-Y_k(x))(y-Y_0(x))}\,.} \tag{5.33}$$

It is easy to check that this expression is a polynomial in y and goes to 0 when $x \to \infty$ in the physical sheet ($\sigma_0(x) \to \infty$) or when $x' \to \infty$ in the physical sheet. It has inverse square root singularities near the endpoints where $\mathcal{X}'(\sigma_k(x)) = 0$.

Computation at $y = Y_0(x)$

$$U(x,Y_0(x);x') = \partial_{x'} \sum_{k=1}^{d_2} \frac{E_y(x,Y_0(x))}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x)-\sigma(x'))(Y_0(x)-Y_k(x))}\,. \tag{5.34}$$

Computation at $x' = x$

$$U(x,Y_0(x);x) = \sum_{k=1}^{d_2} \frac{E_y(x,Y_0(x))}{\mathcal{X}'(\sigma_k(x))\mathcal{X}'(\sigma_0(x))(\sigma_k(x)-\sigma_0(x))^2(Y_0(x)-Y_k(x))}\,. \tag{5.35}$$

We are now equipped to compute the next to leading order functions ...

6. Next to leading order

6.1 The one-loop function: computation of $Y^{(1)}$

We have (see eq. (3.22)):

$$Y^{(1)}(x) = \frac{P^{(1)}(x,Y_0(x))}{E_y(x,Y_0(x))} + \sum_{k=1}^{d_2} \frac{1}{\mathcal{X}'(\sigma_k(x))\mathcal{X}'(\sigma_0(x))(\sigma_k(x)-\sigma_0(x))^2(Y_0(x)-Y_k(x))} \tag{6.1}$$

where $P^{(1)}$ has degrees (d_1-1, d_2-1), and the coefficient of $x^{d_1-1}y^{d_2-1}$ vanishes. We thus have d_1d_2-1 unknown coefficients to determine. On the other hand, there are $2d_1d_2-2$ values of s for which $E_y(\mathcal{X}(s),\mathcal{Y}(s)) = 0$ which are not endpoints (see appendix D, eq. (D.9)). If we write that the poles at all these points cancel in equation 6.1, we can determine $P^{(1)}(x,y)$. However, we don't need to determine $P^{(1)}(x,y)$, we will follow another method which allows to determine $Y^{(1)}(x)$ directly.

Eq. (6.1) is a rational function of $\sigma(x)$, whose only poles are the endpoints, with degree up to 5. Since eq. (6.1) behaves as $O(s^{-2})$ when $s \to \infty$, $Y^{(1)}$ has no pole at ∞, and since eq. (6.1) behaves as $O(s^{1+d_2})$ when $s \to 0$, $\mathcal{X}'(s)Y^{(1)}$ has no pole at 0.

Therefore we may write:

$$Y^{(1)}(\mathcal{X}(s)) = \frac{1}{\mathcal{X}'(s)} \sum_{k=0}^{d_2} \frac{A_k}{(s-e_k)^4} + \frac{B_k}{(s-e_k)^3} + \frac{C_k}{(s-e_k)^2} + \frac{D_k}{(s-e_k)}\,. \tag{6.2}$$

The coefficients A_k, B_k, C_k, D_k are determined by matching the poles in eq. (6.1) ($P^{(1)}$ doesnot contribute to them). We will see below that $D_k = 0$. This is not surprising since we expect $Y^{(1)}(x)$ to be a derivative (indeed $Y^{(1)}(x) = -\partial/\partial V_1(x) F^{(1)} = -d/dx\, \partial/\partial V_1'(x) F^{(1)}$).

Let $k \in [0, d_2]$, and choose s close to e_k:

$$s = e_k + \epsilon \tag{6.3}$$

there must exist $\tilde{s}$ (unique because the potentials are non-critical) such that $\mathcal{X}(\tilde{s}) = \mathcal{X}(s)$ and $\tilde{s}$ is close to e_k ($\tilde{s}$ is the $\sigma_k(x)$ in eq. (6.1)):

$$\tilde{s} = e_k - \eta\,, \qquad \eta = O(\epsilon)\,. \tag{6.4}$$

By solving $\mathcal{X}(s) = \mathcal{X}(\tilde{s})$ order by order in ϵ we get:

$$\eta = \lambda\epsilon\,, \qquad \lambda = 1 + r_k\epsilon + r_k^2\epsilon^2 + (2r_k^3 + t_k)\epsilon^3 + O(\epsilon^4) \tag{6.5}$$

where

$$r_k = \frac{1}{3}\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)}\,, \qquad s_k = \frac{1}{6}\frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)}\,, \qquad t_k = \frac{1}{60}\frac{\mathcal{X}^{V}(e_k)}{\mathcal{X}''(e_k)} - r_k s_k\,. \tag{6.6}$$

From eq. (6.1) we must have:

$$A_k + B_k\epsilon + C_k\epsilon^2 + D_k\epsilon^3 = \frac{\epsilon^4}{\mathcal{X}'(e_k-\eta)(\epsilon+\eta)^2(\mathcal{Y}(e_k+\epsilon) - \mathcal{Y}(e_k-\eta))} + O(\epsilon^4)\,. \tag{6.7}$$

We call $P_k(\epsilon)$ the 3rd degree polynomial:

$$P_k(\epsilon) = A_k + B_k\epsilon + C_k\epsilon^2 + D_k\epsilon^3\,. \tag{6.8}$$

We have:

$$\begin{aligned} P_k(\epsilon) &= (1+\lambda)^{-2}\left(-\lambda\mathcal{X}''(e_k) + \epsilon\frac{\lambda^2}{2}\mathcal{X}'''(e_k) - \epsilon^2\frac{\lambda^3}{6}\mathcal{X}^{IV}(e_k) + \epsilon^3\frac{\lambda^4}{24}\mathcal{X}^{V}(e_k)\right)^{-1} \times \\ &\quad \times\Bigg((1+\lambda)\mathcal{Y}'(e_k) + \frac{\epsilon}{2}(1-\lambda^2)\mathcal{Y}''(e_k) + \frac{\epsilon^2}{6}(1+\lambda^3)\mathcal{Y}'''(e_k) + \\ &\qquad + \frac{\epsilon^3}{24}(1-\lambda^4)\mathcal{Y}^{IV}(e_k)\Bigg)^{-1} + O(\epsilon^4) \\ &= -(\mathcal{X}''(e_k)\mathcal{Y}'(e_k))^{-1}(1+\lambda)^{-3}\lambda^{-1} \times \\ &\quad \times\left(1 - \frac{3}{2}\epsilon\lambda r_k + \epsilon^2\lambda^2 s_k - \frac{5}{2}\epsilon^3(t_k + r_k s_k)\right)^{-1} \times \\ &\quad \times\left(1 + \frac{\epsilon}{2}(1-\lambda)\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} + \frac{\epsilon^2}{6}(1-\lambda+\lambda^2)\frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)}\right)^{-1} + \\ &\quad + O(\epsilon^4)\,. \end{aligned} \tag{6.9}$$

It is easy to see that t_k as well as $\mathcal{Y}^{IV}(e_k)$ disappear, and we find that $D_k = 0$.

After a straightforward calculation, one finds:

$$\begin{aligned} &-8\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(A_k + B_k\epsilon + C_k\epsilon^2 + D_k\epsilon^3) = \\ &\quad = 1 - \frac{1}{3}\epsilon\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} + \frac{1}{6}\epsilon^2\left(\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} + \frac{\mathcal{X}'''(e_k)\mathcal{Y}''(e_k)}{\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)}\right)\,. \end{aligned} \tag{6.10}$$

After substitution into eq. (6.2), we find the genus one correction to the resolvent:

$$
\begin{aligned}
Y^{(1)}(\mathcal{X}(s)) = & -\sum_{k=0}^{d_2} \frac{1}{8\mathcal{X}'(s)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^4} \sum_{k=0}^{d_2} \frac{\mathcal{X}'''(e_k)}{24\mathcal{X}'(s)\mathcal{X}''^2(e_k)\mathcal{Y}'(e_k)(s-e_k)^3} - \\
& -\sum_{k=0}^{d_2} \frac{\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} - \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} + \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)}\frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)}}{48\mathcal{X}'(s)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} .
\end{aligned}
\tag{6.11}
$$

This function represents the partition function of a statistical physics model on a genus one surface with one + boundary.

6.2 The free energy

We are now going to find the free energy $F^{(1)}$ such that:

$$Y^{(1)}(x) = -\frac{\partial F^{(1)}}{\partial V_1(x)} . \tag{6.12}$$

For that purpose, we need to compute the derivatives of various quantities with respect to $V_1(x)$. From our knowledge of the one matrix case [30], we guess that $F^{(1)}$ will be related to the "moments" $\mathcal{Y}'(e_k)$. Therefore, we shall compute the action of $\partial/\partial V_1(x)$ on such quantities.

6.2.1 The derivatives of $\mathcal{Y}(u)$ and $\mathcal{X}(u)$

In this section, we note $x = \mathcal{X}(s)$ and:

$$\dot{\mathcal{X}}(u) = \mathcal{X}'(s)\frac{\partial \mathcal{X}(u)}{\partial V_1(x)} , \qquad \dot{\mathcal{Y}}(u) = \mathcal{X}'(s)\frac{\partial \mathcal{Y}(u)}{\partial V_1(x)} . \tag{6.13}$$

Notice that:

$$\frac{\partial \mathcal{Y}(u)}{\partial V_1(x)} = \frac{\partial Y(\mathcal{X}(u))}{\partial V_1(x)} = \frac{\partial \mathcal{X}(u)}{\partial V_1(x)} Y'(\mathcal{X}(u)) + \left.\frac{\partial Y(x')}{\partial V_1(x)}\right|_{x'=\mathcal{X}(u)} \tag{6.14}$$

which implies (using eq. (5.11)):

$$\mathcal{Y}'(u)\dot{\mathcal{X}}(u) - \mathcal{X}'(u)\dot{\mathcal{Y}}(u) = \frac{1}{(s-u)^2} . \tag{6.15}$$

In particular, if e_k, $k = 0, \ldots, d_2$ is an endpoint $\mathcal{X}'(e_k) = 0$ we have:

$$\dot{\mathcal{X}}(e_k) = \frac{1}{\mathcal{Y}'(e_k)(s-e_k)^2} . \tag{6.16}$$

Now, notice that $\dot{\mathcal{X}}(u)$ is a Laurent Polynomial[9] in u:

$$\dot{\mathcal{X}}(u) = \dot{\gamma} u + \sum_{k=0}^{d_2} \dot{\alpha}_k u^{-k} \tag{6.17}$$

[9] Indeed d_2 is independent of $V_1(x)$. On the contrary, note that $\dot{\mathcal{Y}}(u)$ is not a Laurent polynomial, because the degree of $\mathcal{Y}(u)$ depends on V_1.

where

$$\dot{\gamma} = \mathcal{X}'(s)\frac{\partial\gamma}{\partial V_1(x)}\,, \qquad \dot{\alpha}_k = \mathcal{X}'(s)\frac{\partial\alpha_k}{\partial V_1(x)} \qquad k=0,\dots,d_2\,. \tag{6.18}$$

We know its value at all e_k's, $k=0,\dots,d_2$, therefore (see appendix A):

$$\dot{\mathcal{X}}(u) = u\mathcal{X}'(u)\left[\frac{\dot{\gamma}}{\gamma} + \sum_{k=0}^{d_2}\frac{1}{e_k(s-e_k)^2(u-e_k)\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\right]. \tag{6.19}$$

Since $\alpha_{d_2} = g^*_{d_2+1}\gamma^{d_2}$, by comparing the coefficient of u^1 and u^{-d_2}, we find:

$$\frac{\dot{\gamma}}{\gamma} = \frac{1}{d_2}\frac{\dot{\alpha}_{d_2}}{\alpha_{d_2}} = \frac{1}{2}\sum_{k=0}^{d_2}\frac{1}{(s-e_k)^2e_k^2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} \tag{6.20}$$

and thus:

$$\mathcal{X}'(s)\frac{\partial\mathcal{X}(u)}{\partial V_1(x)} = \dot{\mathcal{X}}(u) = \frac{u\mathcal{X}'(u)}{2}\sum_{k=0}^{d_2}\frac{u+e_k}{u-e_k}\frac{1}{(s-e_k)^2e_k^2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\,. \tag{6.21}$$

Notice that, as a function of s, this expression has double poles at the endpoints e_k.

For any e_i, $i=0,\dots,d_2$, $\mathcal{X}'(e_i)=0$ implies that $\mathcal{X}'(s)\frac{\partial e_i}{\partial V_1(x)} = -\frac{\dot{\mathcal{X}}'(e_i)}{\mathcal{X}''(e_i)}$ and thus:

$$\begin{aligned}\dot{e}_i = \mathcal{X}'(s)\frac{\partial e_i}{\partial V_1(x)} = -\frac{\dot{\mathcal{X}}'(e_i)}{\mathcal{X}''(e_i)} &= -\frac{e_i}{2}\sum_{k\neq i}\frac{e_i+e_k}{e_i-e_k}\frac{1}{(s-e_k)^2e_k^2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} - \\ &\quad -\frac{3+e_i\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}}{2e_i(s-e_i)^2\mathcal{X}''(e_i)\mathcal{Y}'(e_i)}\,.\end{aligned} \tag{6.22}$$

From eq. (6.15) and eq. (6.21) we have:

$$\dot{\mathcal{Y}}(u) = \frac{1}{\mathcal{X}'(u)}\left(\mathcal{Y}'(u)\dot{\mathcal{X}}(u) - \frac{1}{(s-u)^2}\right) \tag{6.23}$$

which implies:

$$\begin{aligned}\mathcal{X}'(s)\frac{\partial\mathcal{Y}'(e_i)}{\partial V_1(x)} &= \dot{\mathcal{Y}}'(e_i) + \dot{e}_i\mathcal{Y}''(e_i) \\ &= \frac{1}{2\mathcal{X}''(e_i)}\Bigg(\dot{\mathcal{X}}(e_i)\mathcal{Y}'''(e_i) + \dot{\mathcal{X}}''(e_i)\mathcal{Y}'(e_i) - \frac{6}{(s-e_i)^4} - \\ &\qquad - \dot{\mathcal{X}}(e_i)\mathcal{Y}''(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)} - \dot{\mathcal{X}}'(e_i)\mathcal{Y}'(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)} + \\ &\qquad + \frac{2}{(s-e_i)^3}\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}\Bigg)\,.\end{aligned} \tag{6.24}$$

This expression has poles of degree 2, 3 and 4, at the e_k's. We write:

$$\mathcal{X}'(s)\frac{\partial\mathcal{Y}'(e_i)}{\partial V_1(x)} = \sum_{k=0}^{d_2}\frac{A_{k,i}}{(s-e_k)^4} + \frac{B_{k,i}}{(s-e_k)^3} + \frac{C_{k,i}}{(s-e_k)^2} \tag{6.25}$$

with:

$$A_{k,i} = -3\frac{\delta_{k,i}}{\mathcal{X}''(e_i)}\,, \qquad B_{k,i} = \frac{\delta_{k,i}\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)^2} \tag{6.26}$$

and:

$$C_{k,i} = \frac{\Big(g_k(e_i)(\mathcal{Y}'''(e_i) - \mathcal{Y}''(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}) - g_k'(e_i)\mathcal{Y}'(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)} + g_k''(e_i)\mathcal{Y}'(e_i)\Big)}{2\mathcal{X}''(e_i)e_k^2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)} \tag{6.27}$$

where

$$g_k(u) = \frac{u\mathcal{X}'(u)}{2}\frac{u+e_k}{u-e_k} \tag{6.28}$$

in particular:

$$g_i(e_i) = e_i^2\mathcal{X}''(e_i)\,, \qquad g_i'(e_i) = \frac{3}{2}e_i\mathcal{X}''(e_i) + \frac{1}{2}e_i^2\mathcal{X}'''(e_i) \tag{6.29}$$

$$g_i''(e_i) = \mathcal{X}''(e_i) + \frac{3}{2}e_i\mathcal{X}'''(e_i) + \frac{1}{3}e_i^2\mathcal{X}^{IV}(e_i) \tag{6.30}$$

and if $k \neq i$:

$$g_k(e_i) = 0\,, \qquad g_k'(e_i) = \frac{e_i}{2}\frac{e_i+e_k}{e_i-e_k}\mathcal{X}''(e_i) \tag{6.31}$$

$$g_k''(e_i) = \frac{e_i+e_k}{e_i-e_k}\mathcal{X}''(e_i) - 2\frac{e_ie_k}{(e_i-e_k)^2}\mathcal{X}''(e_i) + \frac{e_i}{2}\frac{e_i+e_k}{e_i-e_k}\mathcal{X}'''(e_i)\,. \tag{6.32}$$

We thus have:

$$C_{i,i} = \frac{\mathcal{Y}'''(e_i) - \mathcal{Y}''(e_i)\frac{\mathcal{X}'''(e_i)}{\mathcal{X}''(e_i)}}{2\mathcal{X}''(e_i)\mathcal{Y}'(e_i)} - \frac{\mathcal{X}'''(e_i)^2}{4\mathcal{X}''(e_i)^3} + \frac{1}{2e_i^2\mathcal{X}''(e_i)} + \frac{\mathcal{X}^{IV}(e_i)}{6\mathcal{X}''(e_i)^2} \tag{6.33}$$

and if $k \neq i$:

$$C_{k,i} = \frac{(e_i^2 - e_k^2 - 2e_ie_k)\mathcal{Y}'(e_i)}{2e_k^2(e_i-e_k)^2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)}\,. \tag{6.34}$$

Now, let us compute the following quantity (our guess is that it is related to $Y^{(1)}$):

$$\begin{aligned}\mathcal{X}'(s)\sum_{i=0}^{d_2}\frac{\partial\ln\mathcal{Y}'(e_i)}{\partial V_1(x)} &= \sum_{k=0}^{d_2}\frac{A_{k,k}}{\mathcal{Y}'(e_k)(s-e_k)^4} + \sum_{k=0}^{d_2}\frac{B_{k,k}}{\mathcal{Y}'(e_k)(s-e_k)^3} + \\ &\quad + \sum_{k=0}^{d_2}\frac{C_{k,k}}{\mathcal{Y}'(e_k)(s-e_k)^2} + \sum_{k=0}^{d_2}\sum_{i\neq k}\frac{C_{k,i}}{\mathcal{Y}'(e_i)(s-e_k)^2}\,.\end{aligned} \tag{6.35}$$

substituting eqs. (6.26), (6.33), (6.34) into eq. (6.35), we get:

$$\begin{aligned}\mathcal{X}'(s)\sum_{i=0}^{d_2}\frac{\partial\ln\mathcal{Y}'(e_i)}{\partial V_1(x)} &= -3\sum_{k=0}^{d_2}\frac{1}{\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^4} + \sum_{k=0}^{d_2}\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)^2\mathcal{Y}'(e_k)(s-e_k)^3} + \\ &\quad + \sum_{k=0}^{d_2}\frac{\frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)}\frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} - \frac{1}{2}\frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} + \frac{1}{3}\frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)}}{2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} +\end{aligned}$$

$$+\sum_{k=0}^{d_2} \frac{1}{2e_k^2 \mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} +$$
$$+\sum_{k=0}^{d_2} \sum_{i\neq k} \frac{(e_i^2 - e_k^2 - 2e_i e_k)}{2e_k^2 (e_i - e_k)^2 \mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} . \tag{6.36}$$

Notice the following identity:[10]

$$\begin{aligned} 1 + \sum_{i\neq k} \frac{(e_i^2 - e_k^2 - 2e_i e_k)}{(e_i - e_k)^2} &= d_2 + 1 - 2e_k^2 \sum_{i\neq k} \frac{1}{(e_i - e_k)^2} \\ &= -(d_2+1) + \frac{2}{3} e_k^2 \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)} - \frac{e_k^2}{2} \frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} . \end{aligned} \tag{6.37}$$

therefore:

$$\begin{aligned} \mathcal{X}'(s) \sum_{i=0}^{d_2} \frac{\partial \ln \mathcal{Y}'(e_i)}{\partial V_1(x)} &= -3 \sum_{k=0}^{d_2} \frac{1}{\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^4} + \\ &\quad + \sum_{k=0}^{d_2} \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)^2 \mathcal{Y}'(e_k)(s-e_k)^3} + \\ &\quad + \sum_{k=0}^{d_2} \frac{\frac{\mathcal{Y}'''(e_k)}{\mathcal{Y}'(e_k)} - \frac{\mathcal{Y}''(e_k)}{\mathcal{Y}'(e_k)} \frac{\mathcal{X}'''(e_k)}{\mathcal{X}''(e_k)} - \frac{\mathcal{X}'''(e_k)^2}{\mathcal{X}''(e_k)^2} + \frac{\mathcal{X}^{IV}(e_k)}{\mathcal{X}''(e_k)}}{2\mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} - \\ &\quad - (d_2+1) \sum_{k=0}^{d_2} \frac{1}{2e_k^2 \mathcal{X}''(e_k)\mathcal{Y}'(e_k)(s-e_k)^2} . \end{aligned} \tag{6.38}$$

We recognize:

$$\sum_{i=0}^{d_2} \frac{\partial \ln \mathcal{Y}'(e_i)}{\partial V_1(x)} = 24\, Y^{(1)}(x) - (d_2+1) \frac{\partial \ln \gamma}{\partial V_1(x)} . \tag{6.39}$$

6.2.2 The free energy

The free energy is such that:

$$W(x) = \frac{\partial F}{\partial V_1(x)} = V_1'(x) - Y(x) \tag{6.40}$$

therefore to order $1/N^2$ we have:

$$\frac{\partial F^{(1)}}{\partial V_1(x)} = -Y^{(1)}(x) \tag{6.41}$$

which implies (up to a constant K independent of V_1):

$$F^{(1)} = -\frac{1}{24} \ln\left(\gamma^4 D\right) + K \tag{6.42}$$

[10] One proves it by taking the log derivative of $u^{d_2+1}\mathcal{X}'(u) = \gamma \prod_k (u - e_k)$

where:

$$D = \frac{(-1)^{d_1+d_2}}{\gamma^{d_1+d_2+2}} \det \begin{pmatrix} -\gamma & 0 & \alpha_1 & \dots & d_2\alpha_{d_2} & 0 & \dots & 0 \\ 0 & \ddots & & & & \ddots & & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & \dots & 0 & -\gamma & 0 & \alpha_1 & \dots & d_2\alpha_{d_2} \\ d_1\beta_{d_1} & \dots & \beta_1 & 0 & -\gamma & 0 & \dots & 0 \\ 0 & \ddots & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & \dots & 0 & d_1\beta_{d_1} & \dots & \beta_1 & 0 & -\gamma \end{pmatrix} \tag{6.43}$$

D is such that:

$$D = (-1)^{d_2+1} \left(\frac{d_2\alpha_{d_2}}{\gamma}\right)^2 \prod_{\mathcal{X}'(e)=0} \frac{\mathcal{Y}'(e)}{\gamma} = \gamma^2 \prod_{\mathcal{Y}'(\tilde{e})=0} \left(\frac{\mathcal{X}'(\tilde{e})}{\gamma}\right). \tag{6.44}$$

The Constant K is independent of V_1, and if we repeat the calculation by exchanging the roles of M_1 and M_2, we find the same expression for $F^{(1)}$ with K independent of V_2 as well. By comparing with the gaussian case we get $K = 0$.

We have found the genus one free energy:

$$\boxed{F^{(1)} = -\frac{1}{24} \ln\left(\gamma^4 D\right).} \tag{6.45}$$

Remark: D is a determinant which vanishes everytime the surface is singular, i.e. when the potentials are critical. We see that near a critical point, the genus one free energy has a logarithmic divergence. This is in agreement with known double-scaling limit exponents.

6.3 Higher genus expansion

The procedure can be repeated to higher orders. We have the three equations:

$$\begin{aligned} E(x, Y(x)) &= (V_1'(x) - Y(x))(V_2'(Y(x)) - x) - P(x, Y(x)) + 1 \\ &= \frac{1}{N^2} U(x, Y(x); x) \end{aligned} \tag{6.46}$$

$$(y - Y(x))U(x, y) = (V_1'(x) - Y(x))(V_2'(y) - x) - P(x, y) + 1 - \frac{1}{N^2} U(x, y; x) \tag{6.47}$$

$$U(x, y; x') = \frac{\partial U(x, y)}{\partial V_1(x')}. \tag{6.48}$$

Imagine that we already know the functions $Y(x)$, $E(x, y)$ (i.e. $P(x, y)$), $U(x, y)$ and $U(x, y; x')$ up to order $h - 1$. Let us write eq. (6.46) to order h. we know all terms but $Y^{(h)}(x)$ and $P^{(h)}(x, y)$. The factor of the $Y^{(h)}(x)$ term is $E_y^{(0)}(x, Y^{(0)}(x))$, it vanishes for all x such that $E_y^{(0)}(x, Y^{(0)}(x)) = 0$, i.e. at $2d_1d_2 - 2$ points, therefore we know $P^{(h)}(x, y)$ in $2d_1d_2 - 2$ points. Since $P^{(h)}(x, y)$ is a polynomial of degree $(d_1 - 1, d_2 - 1)$, with $d_1d_2 - 1$ unknown coefficients, $P^{(h)}(x, y)$ can be determined completely. Then, from eq. (6.46), $Y^{(h)}(x)$ can be determined. Using eq. (6.47), we can determine $U^{(h)}(x, y)$, and using eq. (6.48) we can determine $U^{(h)}(x, y; x')$ to order h.

We clearly have an algorithmic recursive procedure to find $Y^{(h)}(x)$ to any order h.

Finding the free energy $F^{(h)}$ is more difficult, as one needs to integrate with respect to the potentials. The easiest is to have an ansatz for $F^{(h)}$ and check that its derivative with respect to $V_1(x)$ is indeed $-Y^{(h)}(x)$, as in [30].

Define:

$$M_{0,i} = \mathcal{Y}'(e_i)\,, \qquad M_{k,i} = \frac{\mathcal{Y}^{(k+1)}(e_i)}{\mathcal{Y}'(e_i)} \qquad k \geq 1 \tag{6.49}$$

$$J_{0,i} = \mathcal{X}''(e_i)\,, \qquad J_{k,i} = \frac{\mathcal{X}^{(k+2)}(e_i)}{\mathcal{X}''(e_i)} \qquad k \geq 1\,. \tag{6.50}$$

We expect the genus $h > 1$ free energy to be a rational function of these quantities, of the form:

$$F^{(h)} = \sum_{r,s,p,q,m\geq 0} \sum_{k_1,\ldots,k_s} \sum_{l_1,\ldots,l_s} A_{k_1,\ldots,k_s,l_1,\ldots,l_s,p,q,m} \frac{1}{\gamma^m} \prod_{i=0}^{d_2} \frac{\prod_{j=1}^{r} \prod_{u=1}^{s} M_{k_j,i} J_{l_u,i}}{M_{0,i}^p J_{0,i}^q} \tag{6.51}$$

with the restrictions that:

$$p + q = \sum_j k_j + \sum_u l_u\,. \tag{6.52}$$

The selection rules (of the form $m + p + q \leq 4h - 4$ as in [30]) are still to be understood.

7. The one matrix case: $d_2 = 1$

In this section we check that for $d_2 = 1$, our results reduce to those of [30].

Consider $d_2 = 1$ and $V_2(y) = \frac{1}{2} g_2^* y^2 + g_1^* y$. V_2 is thus quadratic and M_2 can easily be integrated out. The 2-matrix model in this case reduces to a one matrix model with the potential:

$$V(x) = V_1(x) - \frac{1}{2g_2^*} x^2 + \frac{g_1^*}{g_2^*} x\,. \tag{7.1}$$

The function $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ are:

$$\mathcal{X}(s) = \gamma s + \alpha_0 + \frac{\alpha_1}{s}\,, \qquad \mathcal{Y}(s) = \frac{\gamma}{s} + \sum_{k=0}^{d_1} \beta_k s^k\,. \tag{7.2}$$

From eq. (4.4) we find that:

$$\alpha_1 = g_2^* \gamma\,, \qquad \alpha_0 = g_2^* \beta_0 + g_1^* \tag{7.3}$$

and:

$$V_1'(\mathcal{X}(s)) = \mathcal{Y}(s) + \mathcal{Y}\left(\frac{g_2^*}{s}\right) - \beta_0 - \frac{\gamma}{s} - \frac{\gamma}{g_2^*} s \tag{7.4}$$

indeed this expression is a polynomial in $x = \mathcal{X}(s)$ (because $\mathcal{X}\left(g_2^*/s\right) = x$) and it satisfies eq. (4.4). It follows from eq. (7.1):

$$V'(\mathcal{X}(s)) = \mathcal{Y}(s) + \mathcal{Y}\left(\frac{g_2^*}{s}\right) - 2\left(\beta_0 + \frac{\gamma}{s} + \frac{\gamma}{g_2^*} s\right). \tag{7.5}$$

JHEP01(2003)051

The endpoints a and b are the roots of $\mathcal{X}'(s)$ i.e. they correspond to:

$$e = \sqrt{g_2^*}\,, \qquad a = \mathcal{X}(-e)\,, \qquad b = \mathcal{X}(+e)\,, \qquad \frac{b-a}{4} = \gamma\sqrt{g_2^*}\,. \tag{7.6}$$

The resolvent (defined in eq. (2.6)) can be written [29]:

$$W(x) = V_1'(x) - Y(x) = \frac{1}{2}\left(V'(x) - M(x)\sqrt{(x-a)(x-b)}\right) \tag{7.7}$$

where $M(x)$ is a polynomial in x (the same as in [29]). Using eq. (7.5) and eq. (7.2) we find that:

$$M(\mathcal{X}(s)) = \frac{\mathcal{Y}(s) - \mathcal{Y}(g_2^*/s)}{s\mathcal{X}'(s)}\,, \qquad \sqrt{(\mathcal{X}(s)-a)(\mathcal{X}(s)-b)} = s\mathcal{X}'(s)\,. \tag{7.8}$$

In particular at the endpoints we have:

$$M(a) = \frac{\mathcal{Y}'(-e)}{\gamma}\,, \qquad M(b) = \frac{\mathcal{Y}'(+e)}{\gamma}\,. \tag{7.9}$$

It is clear now, that our result eq. (6.45) coincides with the genus one free energy found by the authors of [29]:

$$F^{(1)} = -\frac{1}{24}\ln\left(\left(\frac{b-a}{4}\right)^4 M(a)M(b)\right). \tag{7.10}$$

8. Conclusion and prospects

The recursive method presented in this article, allows one to compute the resolvents as well as other loop-functions, to any order in the topological expansion of the 2-matrix model. We have found the expression of the genus one free energy, but an algorithmic method to compute the free energy to higher orders is still incomplete. For that purpose, it would be usefull to define some "moments" and basis functions, as in [30], onto which one could decompose the loop-functions, and selection rules to know what set of moments should appear in the free energy at a given order.

It would be interesting also to understand what terms survive in the double scaling limit when the potentials are near a critical point, and to develop a recursive method to compute directly the free energy and resolvents in the double scaling limit.

As we said in the introduction, we have only developped a method to compute the terms in the $1/N^2$ expansion, but we have not proven the existence of that expansion. A rigorous proof of that existence can probably be obtained by Riemann-Hilbert technics [22].

It would also be usefull to understand what the $1/N^2$ expansion becomes in the non genus-zero case. It is very likely that it would involve the θ-functions associated to the algebraic curve, as in [41].

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A. The interpolation formula

Let d be an integer, and $x_0, \ldots, x_d$ be $d+1$ distinct complex numbers. We note:

$$\pi(x) = \prod_{k=0}^{d} (x - x_k)\,. \tag{A.1}$$

Let $P_0, \ldots, P_d$ be $d+1$ complex numbers.

There is a unique polynomial P of degree d such that:

$$P(x_k) = P_k \qquad k = 0, \ldots, d\,. \tag{A.2}$$

That polynomial is:

$$P(x) = \sum_{k=0}^{d} P_k \frac{\pi(x)}{(x - x_k)\pi'(x_k)}\,. \tag{A.3}$$

Any polynomial $P(x)$ of degree $> d$ such that:

$$P(x_k) = P_k \qquad k = 0, \ldots, d \tag{A.4}$$

can be written:

$$P(x) = Q(x)\pi(x) + \sum_{k=0}^{d} P_k \frac{\pi(x)}{(x - x_k)\pi'(x_k)} \tag{A.5}$$

where $Q(x)$ is a polynomial of degree $\deg P - d - 1$.

B. Proof that $P(x, y; x')$ is a polynomial of degree $d_2 - 1$

The leading term of eq. (5.24) is (we use eq. (5.28)):

$$\partial x' \frac{1}{x - x'} \sum_{k=0}^{d_2} \left[\frac{E(x', Y_k(x))}{(Y_k(x) - Y(x'))} \frac{1}{E_y(x, Y_k(x))} - \frac{x - x'}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x) - \sigma(x'))} \right]. \tag{B.1}$$

We have (see eq. (D.5)):

$$\sum_{k=0}^{d_2} \frac{1}{\mathcal{X}'(\sigma_k(x))(\sigma_k(x) - \sigma(x'))} = \frac{1}{x - x'}\,. \tag{B.2}$$

Therefore we have to prove that:

$$\sum_{k=0}^{d_2} \frac{E(x', Y_k(x))}{(Y_k(x) - Y(x'))} \frac{1}{E_y(x, Y_k(x))} = 1\,. \tag{B.3}$$

Note that:

$$\sum_{k=0}^{d_2} \frac{E(x', Y_k(x))}{(Y_k(x) - Y(x'))} \frac{1}{E_y(x, Y_k(x))} = \sum_{k=0}^{d_2} \frac{\prod_{j\neq 0}(Y_k(x) - Y_j(x'))}{\prod_{j\neq k}(Y_k(x) - Y_j(x))}\,. \tag{B.4}$$

Lemma. *For any* $Y_0,\dots,Y_{d_2}$ *and for any* $Y'_1,\dots,Y'_{d_2}$ *we have:*

$$\sum_{k=0}^{d_2} \frac{\prod_{j\neq 0}(Y_k - Y'_j)}{\prod_{j\neq k}(Y_k - Y_j)} = 1\,. \tag{B.5}$$

Proof.

$$\sum_{k=0}^{d_2} \frac{\prod_{j\neq 0}(Y_k - Y'_j)}{\prod_{j\neq k}(Y_k - Y_j)} = \sum_{k=0}^{d_2}(-1)^k f(Y_k)\frac{\Delta_{d_2}(Y_0,\dots,Y_{k-1},Y_{k+1},\dots,Y_{d_2})}{\Delta_{d_2+1}(Y_0,\dots,Y_{d_2})} \tag{B.6}$$

where $f(Y) = \prod_{j\neq 0}(Y - Y'_j)$ is a monic polynomial of degree d_2 in Y, and Δ_n is the Vandermonde determinant of n variables.

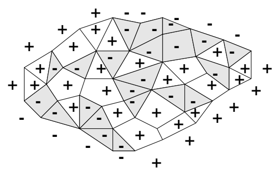
We recognize a determinant (which is 1 by linearly combining columns):

$$\sum_{k=0}^{d_2} \frac{\prod_{j\neq 0}(Y_k - Y'_j)}{\prod_{j\neq k}(Y_k - Y'_j)} = \frac{\det\begin{pmatrix} 1 & Y_0 & Y_0^2 & \dots & Y_0^{d_2-1} & f(Y_0) \\ 1 & Y_1 & Y_1^2 & \dots & Y_1^{d_2-1} & f(Y_1) \\ \vdots & & & & & \vdots \\ 1 & Y_0 & Y_{d_2}^2 & \dots & Y_{d_2}^{d_2-1} & f(Y_{d_2}) \end{pmatrix}}{\Delta_{d_2+1}(Y_0,\dots,y_{d_2})} = 1\,. \tag{B.7}$$

C. Other loop-functions

Define:

$$\begin{aligned} H(x,y,x',y') &= \frac{1}{2N}\left\langle \mathrm{tr}\,\frac{1}{x-M_1}\frac{1}{y-M_2}\frac{1}{x'-M_1}\frac{1}{y'-M_2}\right\rangle + \\ &\quad + \frac{1}{2N}\left\langle \mathrm{tr}\,\frac{1}{y'-M_2}\frac{1}{x'-M_1}\frac{1}{y-M_2}\frac{1}{x-M_1}\right\rangle \end{aligned}$$

$=$  . (C.1)

Notice that we have:

$$H(x,y,x',y') = H(x',y',x,y) = H(x,y',x',y)\,. \tag{C.2}$$

We also define:

$$\begin{aligned} F(x,y,x',y') &= \frac{1}{2N}\left\langle \mathrm{tr}\,\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{1}{y-M_2}\frac{1}{x'-M_1}\frac{1}{y'-M_2}\right\rangle + \\ &\quad + \frac{1}{2N}\left\langle \mathrm{tr}\,\frac{1}{y'-M_2}\frac{1}{x'-M_1}\frac{1}{y-M_2}\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\right\rangle \end{aligned} \tag{C.3}$$

we have:

$$F(x,y,x',y') = F(x,y',x',y)\,. \tag{C.4}$$

Loop equations: the invariance under the change of variable

$$\delta M_1 = \frac{1}{x-M_1}\frac{1}{y-M_2}\frac{1}{x'-M_1}\frac{1}{y'-M_2} + \text{h.c.}$$

implies:

$$(y'-Y(x))H(x,y,x',y') = -\frac{W(x,y)-W(x',y)}{x-x'}(1-W(x',y')) - F(x,y,x',y')\,. \quad \text{(C.5)}$$

The same equation with $y \leftrightarrow y'$ gives:

$$\begin{aligned} H(x,y,x',y') \;=\; & \frac{1}{(x-x')(y-y')}\left[\frac{E(x',y)E(x,y')}{(x'-X)(y-Y')(x-X')(y'-Y)} - \right. \\ & \left. - \frac{E(x,y)E(x',y')}{(x-X)(y-Y)(x'-X')(y'-Y')}\right] \end{aligned} \quad \text{(C.6)}$$

it follows:

$$\begin{aligned} F(x,y,x',y') \;=\; & \frac{E(x,y)E(x',y')}{(x-x')(y-y')(x-X)(x'-X')(y'-Y')} - \\ & - \frac{E(x',y)E(x,y')}{(x-x')(y-y')(x-X')(x'-X)(y-Y')} - \\ & - \frac{E(x',y)E(x',y')}{(x-x')(x'-X)(x'-X')(y-Y')(y'-Y')}\,. \end{aligned} \quad \text{(C.7)}$$

D. Some relations

The following formula are usefull for many calculations in this article. They are easy to prove.

$$E(x,y) = A\prod_{i,j}(\sigma_i(x)-\tilde\sigma_j(y)) \quad \text{(D.1)}$$

with

$$A = (-1)^{d_2+1}\gamma^{1+d_1-d_2}\beta_{d_1}^{d_2}g_{d_1+1} = (-1)^{d_2+1}\frac{\beta_{d_1}^{d_2+1}}{\gamma^{d_2-1}} \quad \text{(D.2)}$$

$$\mathcal{X}(s) - x = \frac{\gamma}{s^{d_2}}\prod_{i=0}^{d_2}(s-\sigma_i(x))\,, \qquad \mathcal{Y}(s) - y = \frac{\beta_{d_1}}{s}\prod_{j=0}^{d_1}(s-\tilde\sigma_j(y)) \quad \text{(D.3)}$$

$$\mathcal{X}'(\sigma_k(x)) = \frac{\gamma}{\sigma_k^{d_2}(x)}\prod_{i\neq k}(\sigma_k(x)-\sigma_i(x))\,, \qquad \mathcal{Y}'(\tilde\sigma_k(y)) = \frac{\beta_{d_1}}{\tilde\sigma_k(y)}\prod_{j\neq k}(\tilde\sigma_k(y)-\tilde\sigma_j(y)) \quad \text{(D.4)}$$

$$\begin{aligned} \sum_{i=0}^{d_2}\frac{1}{\mathcal{X}'(\sigma_i(x))(s-\sigma_i(x))} &= \frac{1}{\mathcal{X}(s)-x} \\ \sum_{j=0}^{d_1}\frac{1}{\mathcal{Y}'(\tilde\sigma_j(y))(s-\tilde\sigma_j(y))} &= \frac{1}{\mathcal{Y}(s)-y} \end{aligned} \quad \text{(D.5)}$$

$$
\begin{aligned}
E_y(x, Y_0(x)) &= -A \frac{1}{\mathcal{Y}'(\sigma_0(x))} \prod_{(i,j)\neq(0,0)} (\sigma_i(x) - \tilde{\sigma}_j(Y_0(x))) \\
&= \frac{\beta_{d_1}^{d_2}}{\gamma^{d_2}} \sigma_0^{d_2+1}(x) \mathcal{X}'(\sigma_0(x)) \prod_{i\neq 0} \prod_{j\neq 0} (\sigma_i(x) - \tilde{\sigma}_j(Y_0(x)))
\end{aligned} \tag{D.6}
$$

$$
Y_k(x) - Y_0(x) = \frac{\beta_{d_1}}{\sigma_k(x)} \prod_j (\sigma_k(x) - \tilde{\sigma}_j(Y_0(x)))\,. \tag{D.7}
$$

Limits:

$$
\begin{aligned}
E_y(\mathcal{X}(s), \mathcal{Y}(s)) &\sim_{s\to\infty} -\frac{\alpha_{d_2}\beta_{d_1}^{d_2}}{\gamma^{d_2}} s^{d_1 d_2} \\
E_y(\mathcal{X}(s), \mathcal{Y}(s)) &\sim_{s\to 0} -d_2 \frac{\beta_{d_1}\alpha_{d_2}^{d_1+1}}{\gamma^{d_1+1}} s^{1-d_1 d_2 - d_2}\,.
\end{aligned} \tag{D.8}
$$

$E_y(\mathcal{X}(s), \mathcal{Y}(s))$ is a Laurent polynomial:

$$
E_y(\mathcal{X}(s), \mathcal{Y}(s)) = -\frac{\mathcal{X}'(s)}{s^{2-d_1 d_2}} P_{2d_1 d_2 - 2}(s) \tag{D.9}
$$

where $P_{2d_1d_2-2}(s)$ is a polynomial of degree $2d_1d_2 - 2$. Therefore, $E_y(\mathcal{X}(s), \mathcal{Y}(s)) = 0$ has $2d_1d_2 - 2$ zeroes.

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